A Differential View of Network Capacity

International Zurich Seminar - February 2018

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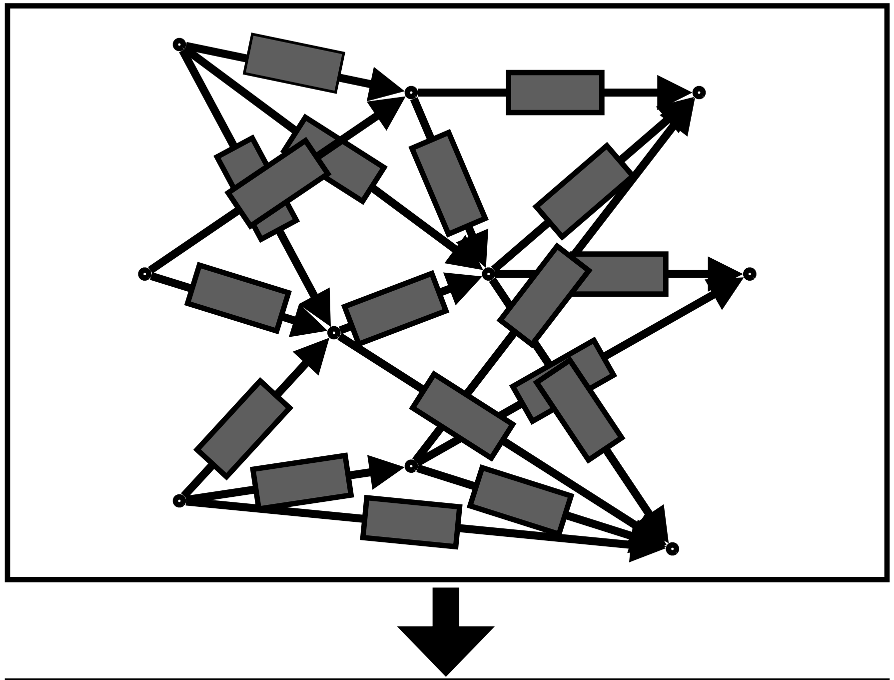
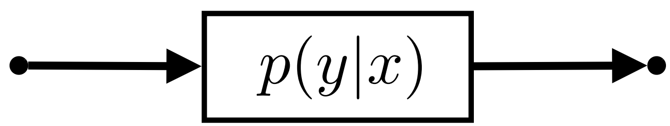
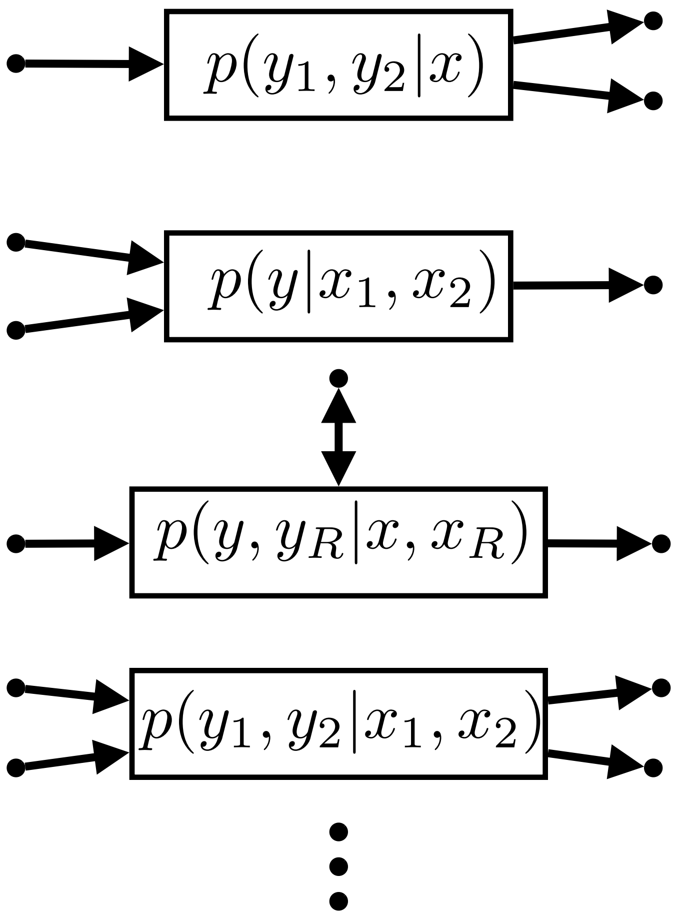
Shannon took an "elementary" approach to network capacity.





$$C = \max_{p(x)} I(X;Y)$$

To a large extent,
the elementary approach persists in the literature.



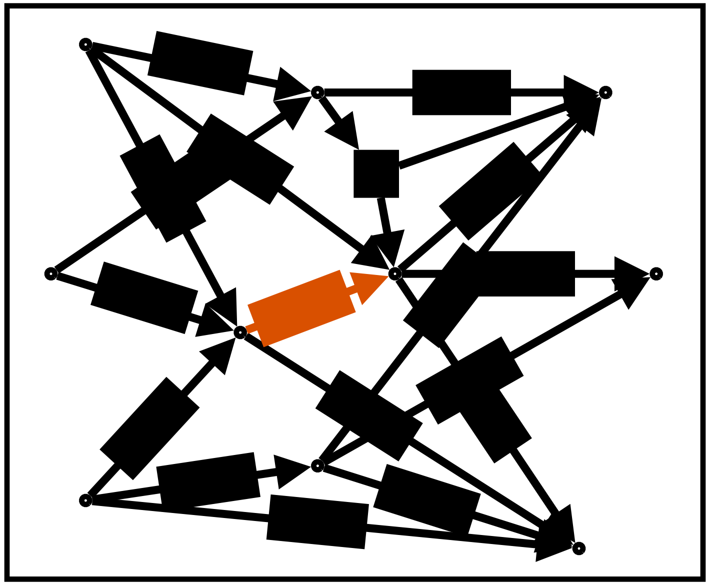
$$\mathcal{C} \subseteq \overline{\{(R_1, R_2) : \exists p(x, u) \text{ s.t. } R_1 < I(X; Y_1|U), R_2 < I(U; Y_2)\}}$$

$$\mathcal{C} = \overline{\{(R_1, R_2) : \exists p(x_1)p(x_2) \text{ s.t. } R_1 < I(X_1; Y|X_2), R_2 < I(X_2; Y|X_1) \; R_1 + R + 2 < I(X_1, X_2; Y)\}}$$

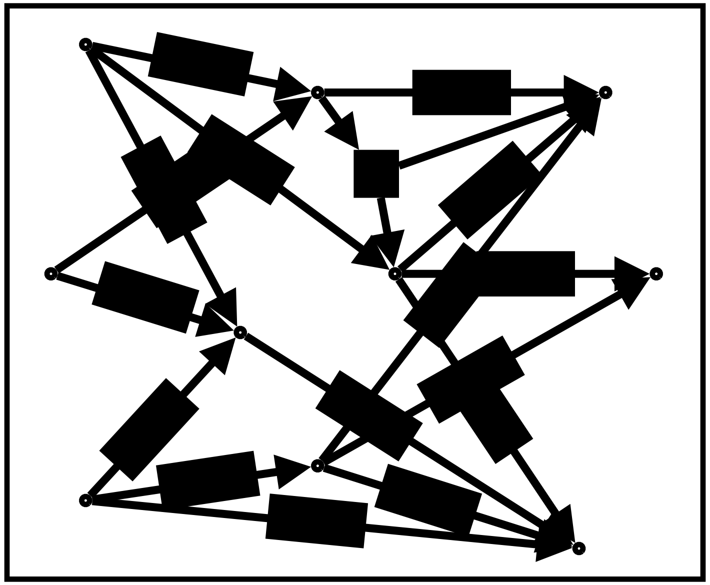
$$\mathcal{C} \subseteq \overline{\{R : \exists p(x, x_R) \text{ s.t. } R < \min\{I(X; Y_R|X_R), I(X, X_R; Y)\}}$$

⋮

We consider a differential approach.



$\mathcal{N} + \mathcal{C}$

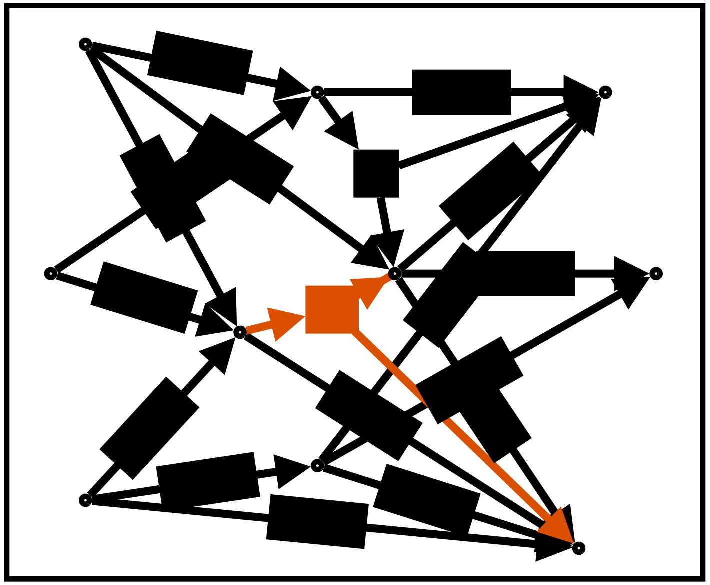


$\mathcal{N}$

$$\text{Find the smallest set } \Delta(\mathcal{C}) \text{ s.t.} \forall\, \mathcal{N}$$

$$\text{Capacity}(\mathcal{N} + \mathcal{C}) \subseteq \text{Capacity}(\mathcal{N}) + \Delta(\mathcal{C})$$

The differential approach is applicable for any channel $\mathcal{C}$.



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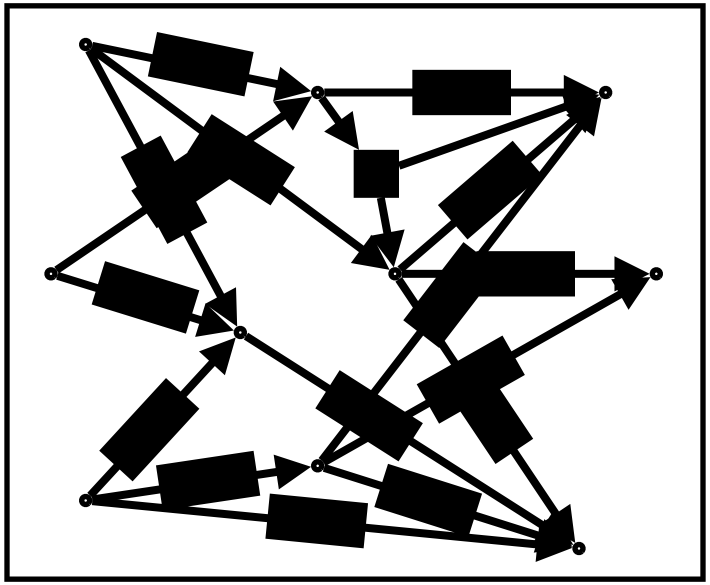

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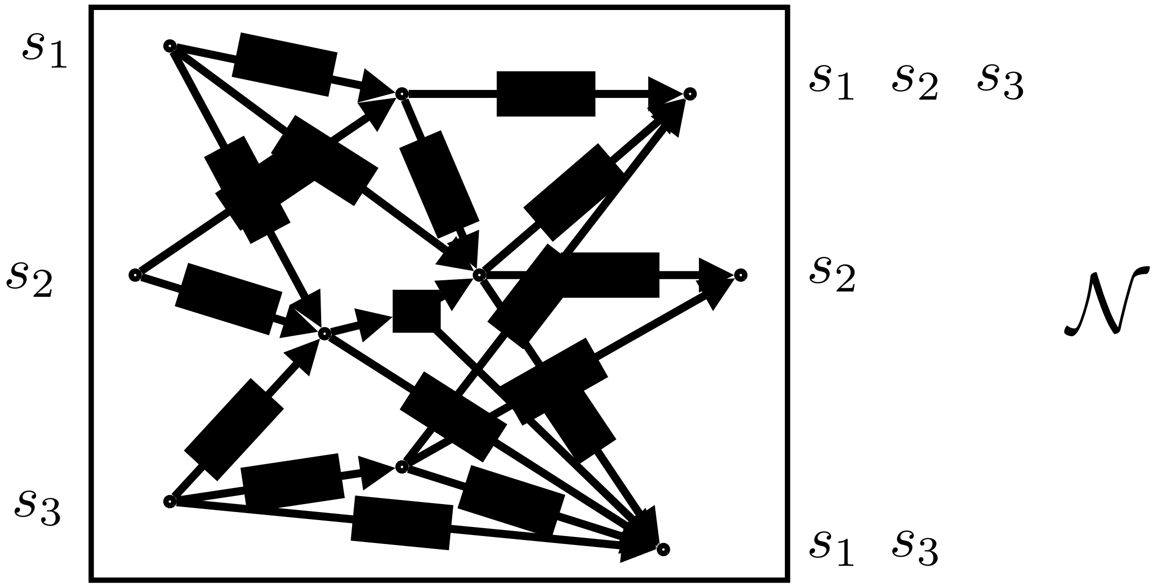
CAPACITY



$$\mathcal{N} = (p, S, D) : \text{ Network}$$

$$p = \text{ Memoryless channel (noisy or noiseless)}$$

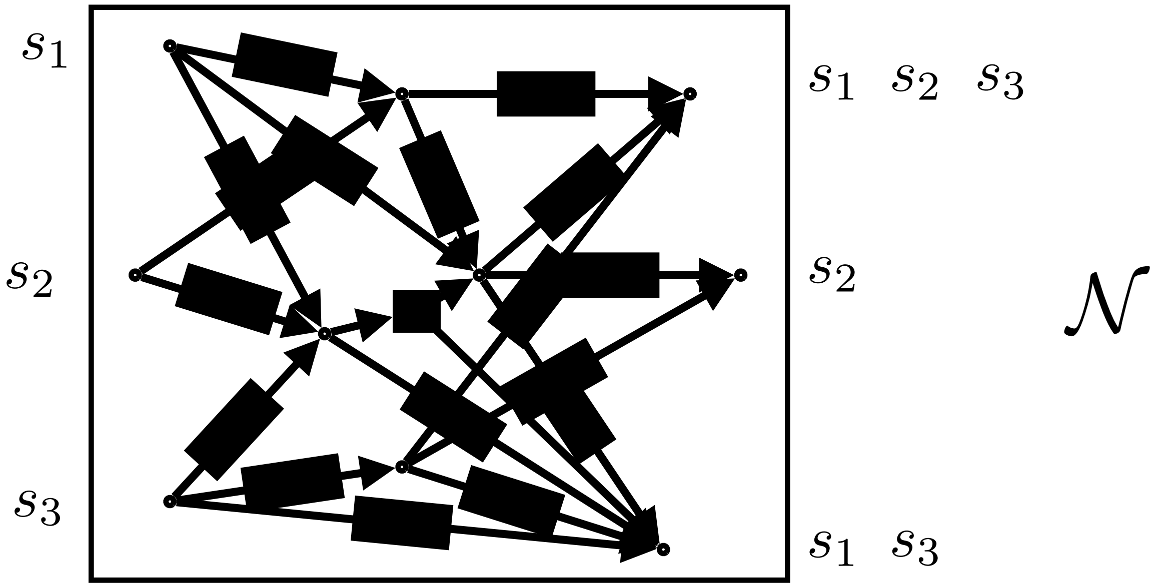
$$S = \{s_1, \ldots, s_k\} : \; s_i = \text{origin of source } i$$

Messages $W_1, \ldots, W_k$ indep, uniform

$$D = \{D(s) : s \in S\} : \text{ demand set}$$

$D(s) =$ nodes that demand the source from node s

CAPACITY



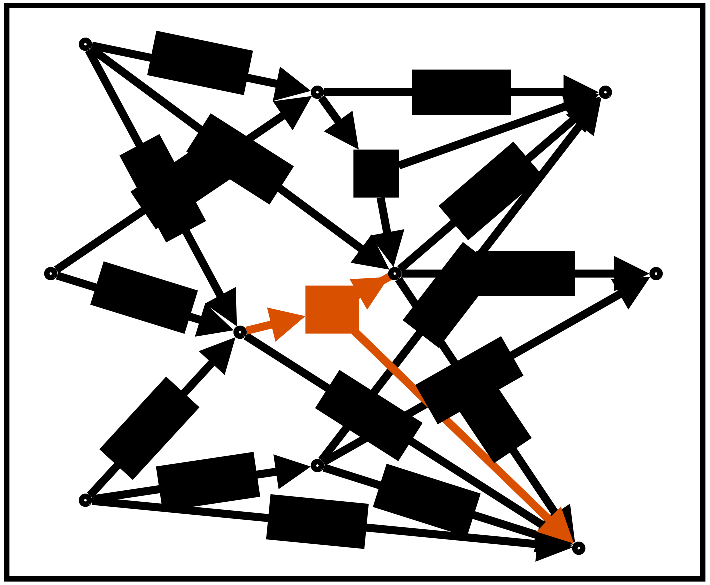
$(R_1, \ldots, R_k)$ achievable

$\Leftrightarrow \quad \exists$ sequence of $((2^{nR_1}, \ldots, 2^{nR_k}), n)$ codes

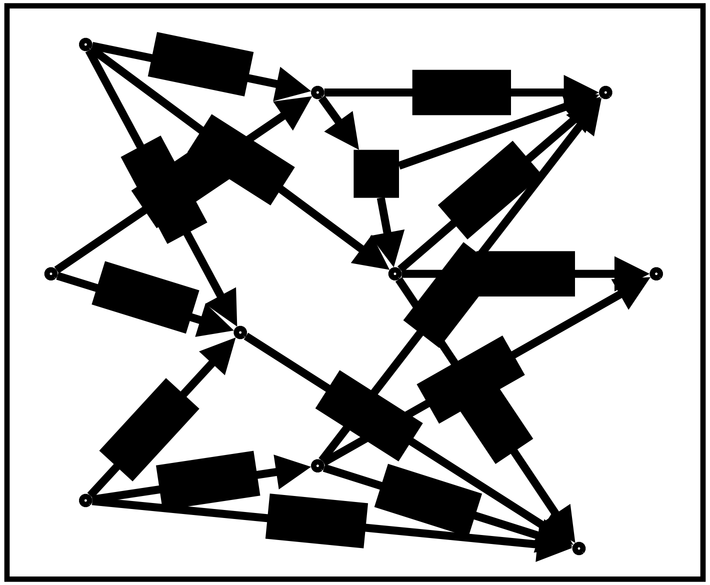
with $P_e^{(n)} \to 0$ as $n \to \infty$

$$\text{Capacity}(\mathcal{N}) = \overline{\{(R_1, \ldots, R_k) : (R_1, \ldots, R_k) \text{ achievable}\}}$$

We consider a differential approach.



$\mathcal{N} + \mathcal{C}$



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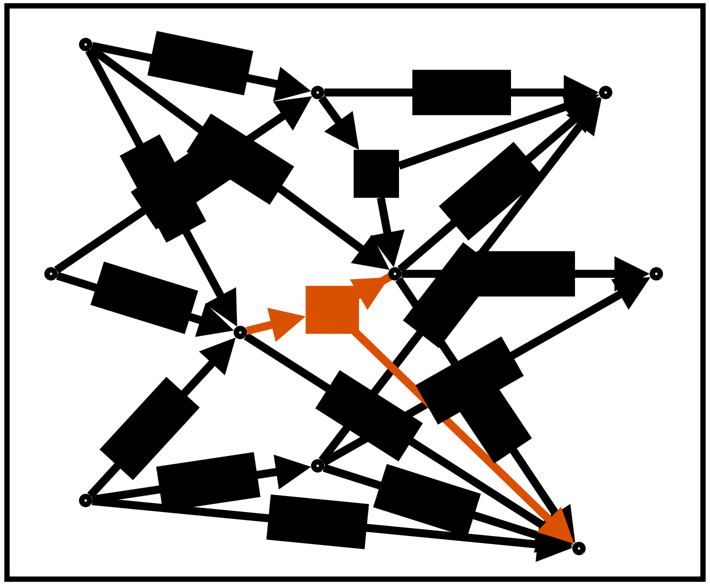
$$\text{Capacity}(\mathcal{N} + \mathcal{C}) \subseteq \text{Capacity}(\mathcal{N}) + \Delta(\mathcal{C})$$

How much can we gain by adding a single channel?

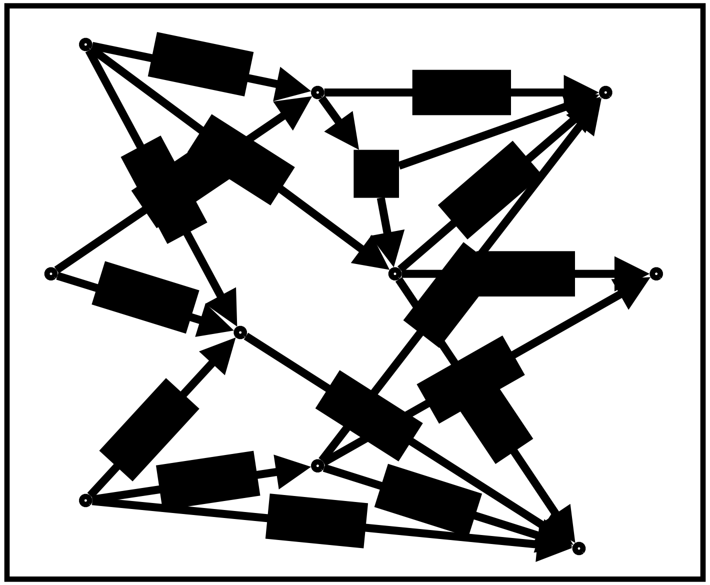
How **vulnerable** are our networks to failures?

What is the impact of each channel --
in the larger context in which it will be employed?

How much do we need to know about a channel to understand its differential impact?



$\mathcal{N} + \mathcal{C}$

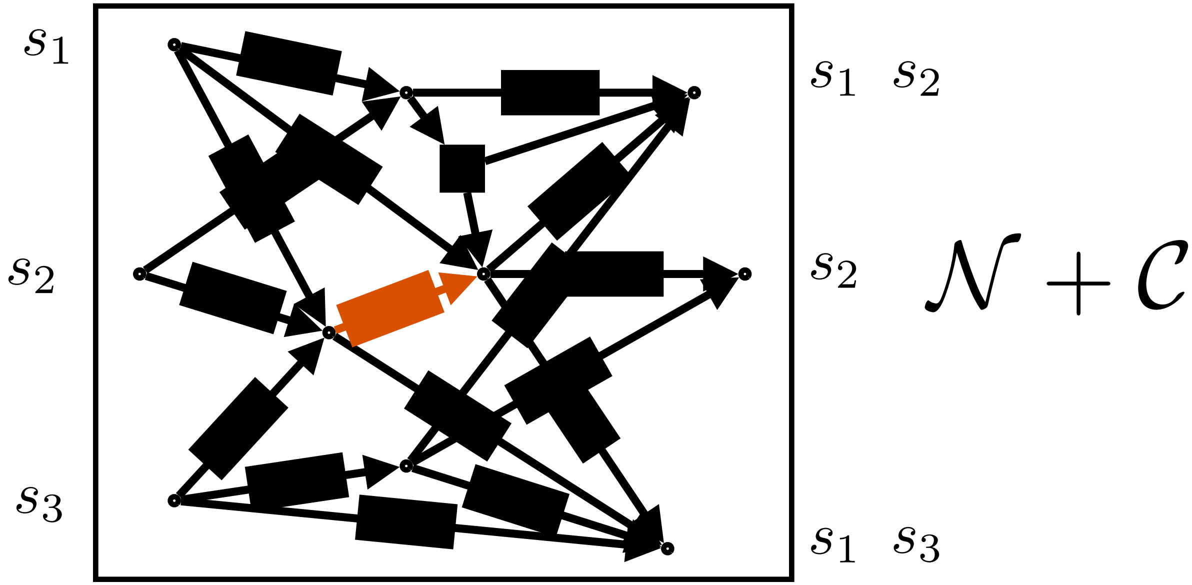


$\mathcal{N}$

$$\text{Find the smallest set } \Delta(\mathcal{C}) \text{ s.t.} \forall\ \mathcal{N}$$

$$\text{Capacity}(\mathcal{N} + \mathcal{C}) \subseteq \text{Capacity}(\mathcal{N}) + \Delta(\mathcal{C})$$

Does a link's capacity determine its differential impact?



Yes! For any network $\mathcal{N}$ and any point-to-point channels $\mathcal{C}$ and $\mathcal{C}'$,

if Capacity($\mathcal{C}$)=Capacity($\mathcal{C}'$)

then Capacity$(\mathcal{N} + \mathcal{C})$ = Capacity$(\mathcal{N} + \mathcal{C}')$

[Koetter, Effros, Medard 2009, 2011]

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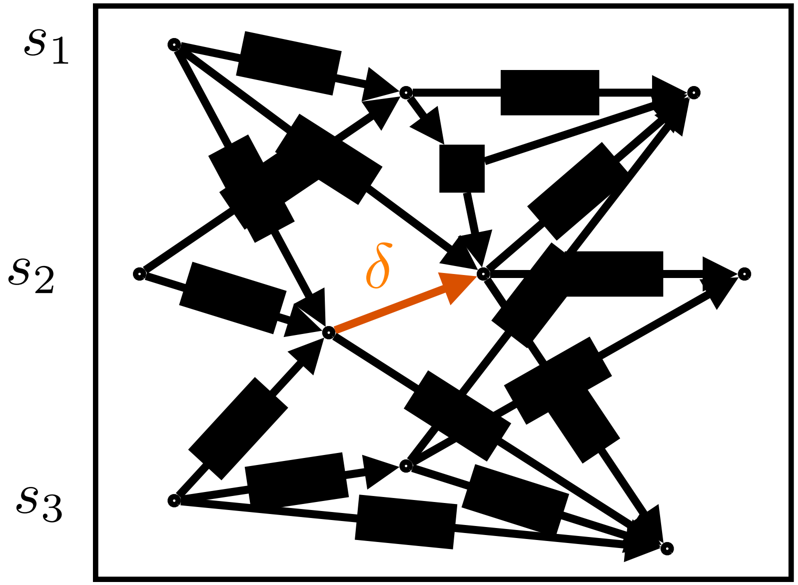


$\mathcal{N} + \delta$

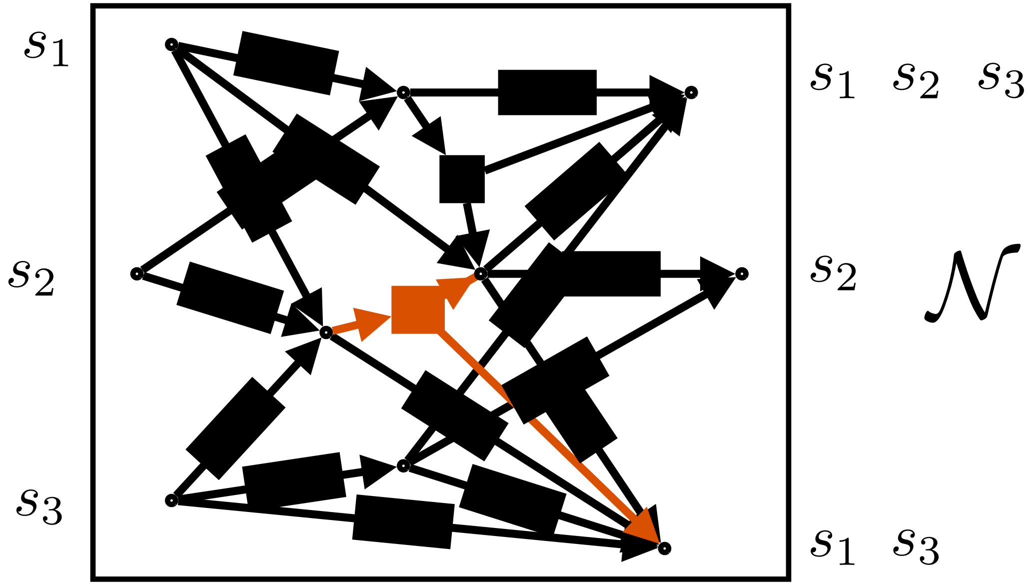
Yes! For any network $\mathcal{N}$ and any point-to-point channels $\mathcal{C}$ and $\mathcal{C}'$,

if Capacity($\mathcal{C}$)=Capacity($\mathcal{C}'$)

then Capacity($\mathcal{N} + \mathcal{C}$) = Capacity($\mathcal{N} + \mathcal{C}'$)

[Koetter, Effros, Medard 2009, 2011]

Does channel capacity determine differential channel impact in general?



$\mathcal{N} + \mathcal{C}$

No! For any network $\mathcal{N}$ and any channels $\mathcal{C}$ and $\mathcal{C}'$,

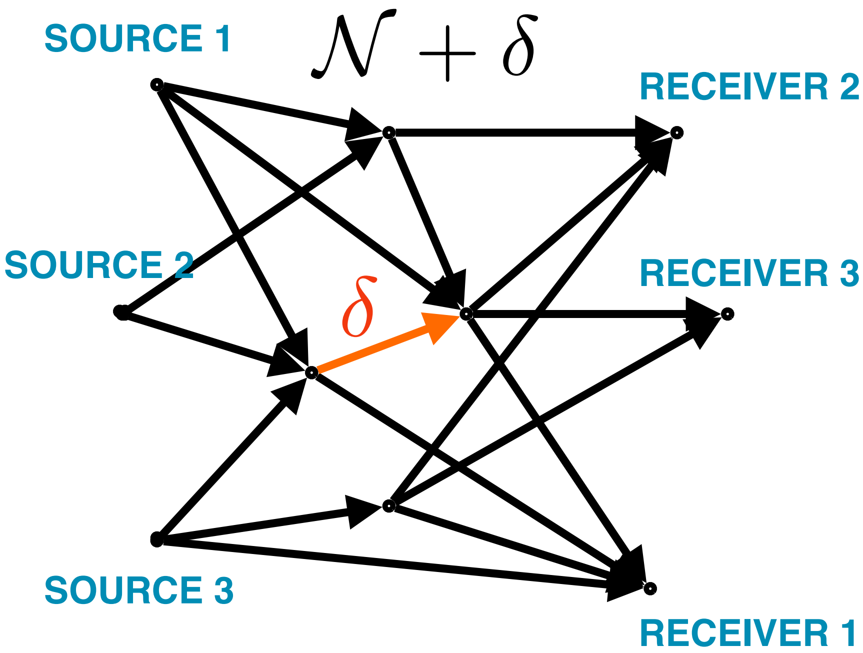
$$\text{Capacity}(\mathcal{C}) = \text{Capacity}(\mathcal{C}')$$

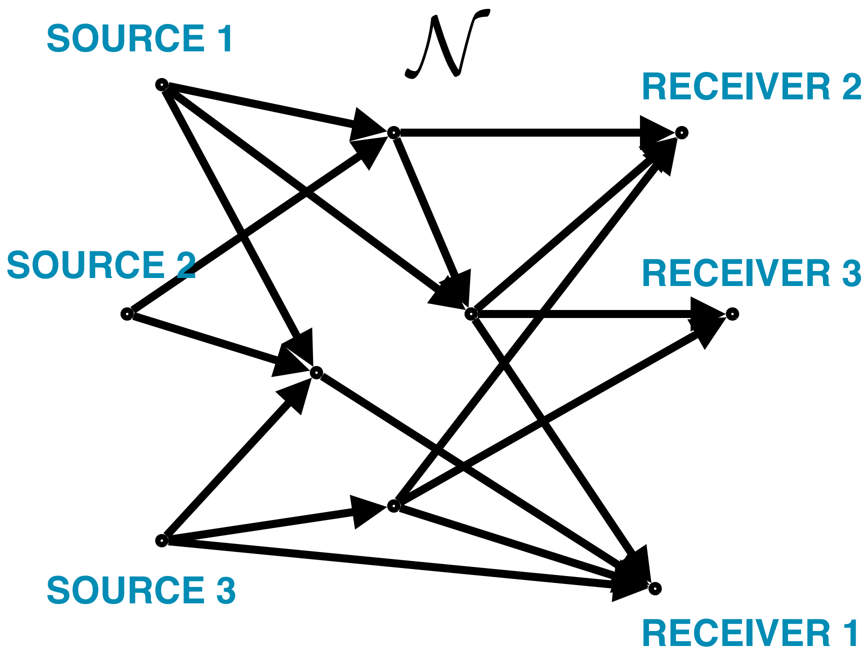
does not imply $\text{Capacity}(\mathcal{N} + \mathcal{C}) = \text{Capacity}(\mathcal{N} + \mathcal{C}')$.

[Koetter, Effros, Medard 2009, 2011]

To begin, consider the impact of a wireline link in a network of wireline links.

What can we say about a link's differential impact in a **wireline network**?





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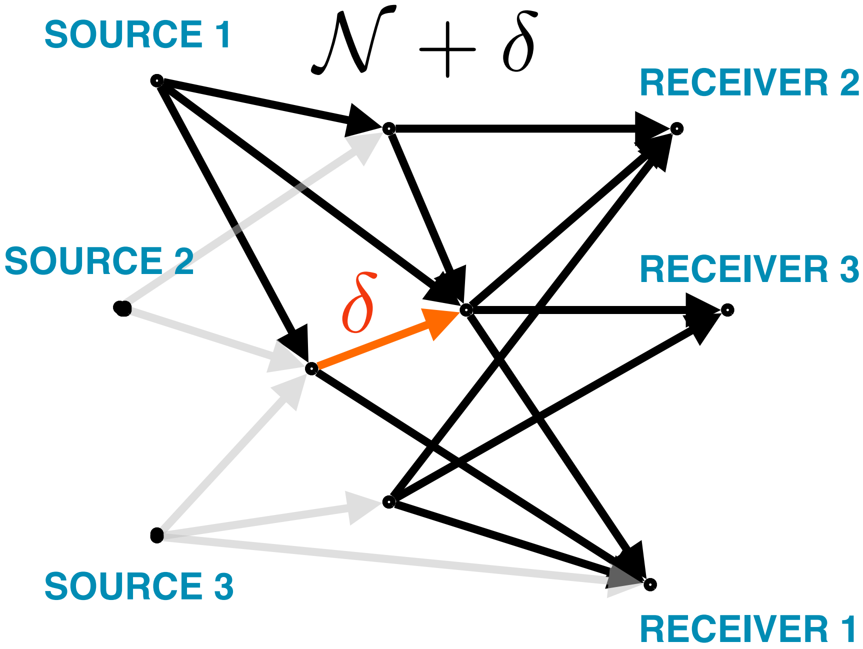

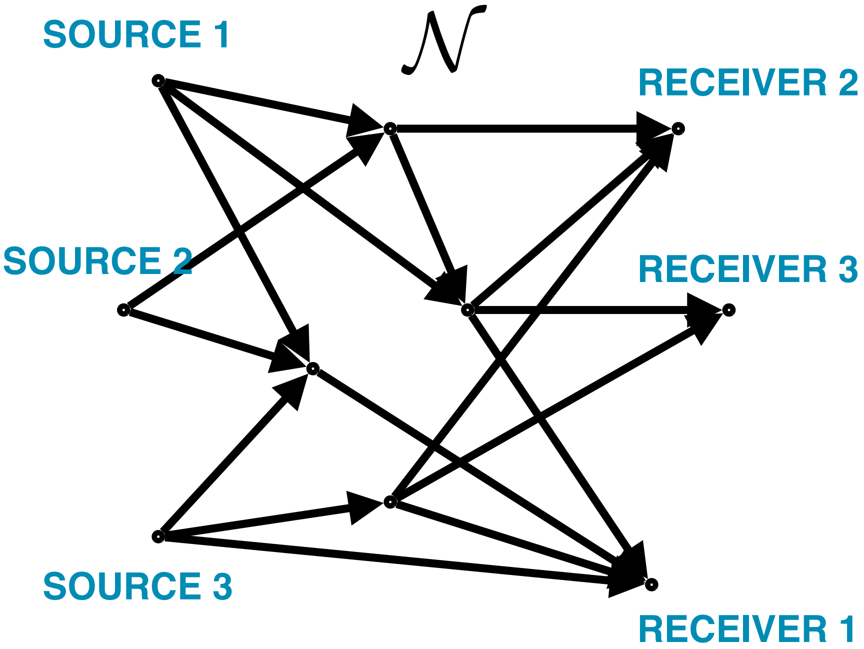

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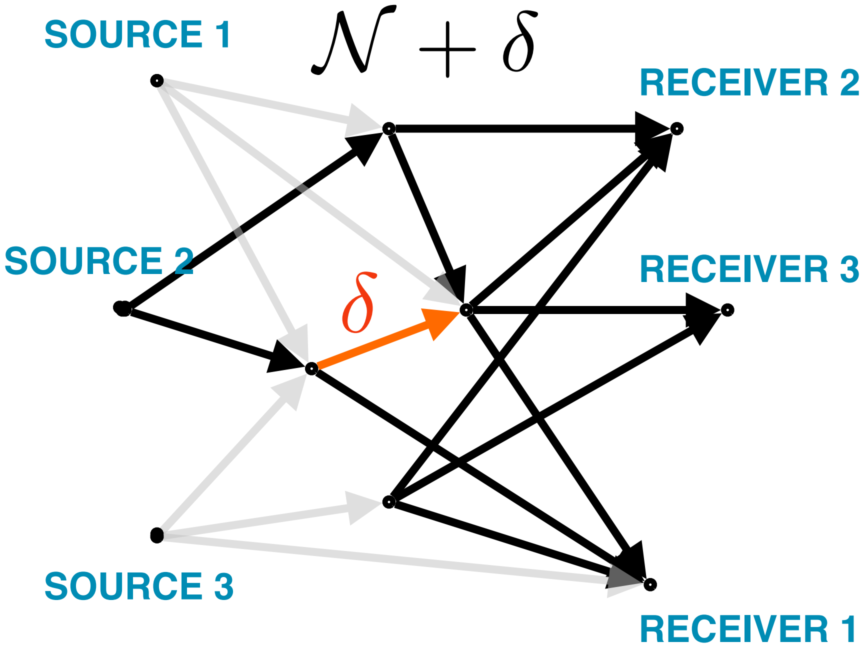

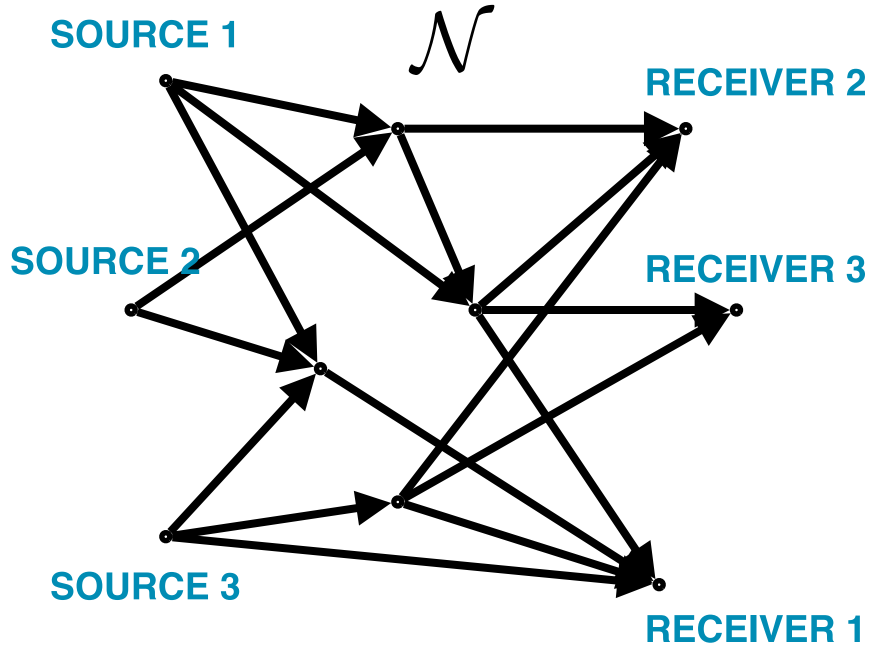

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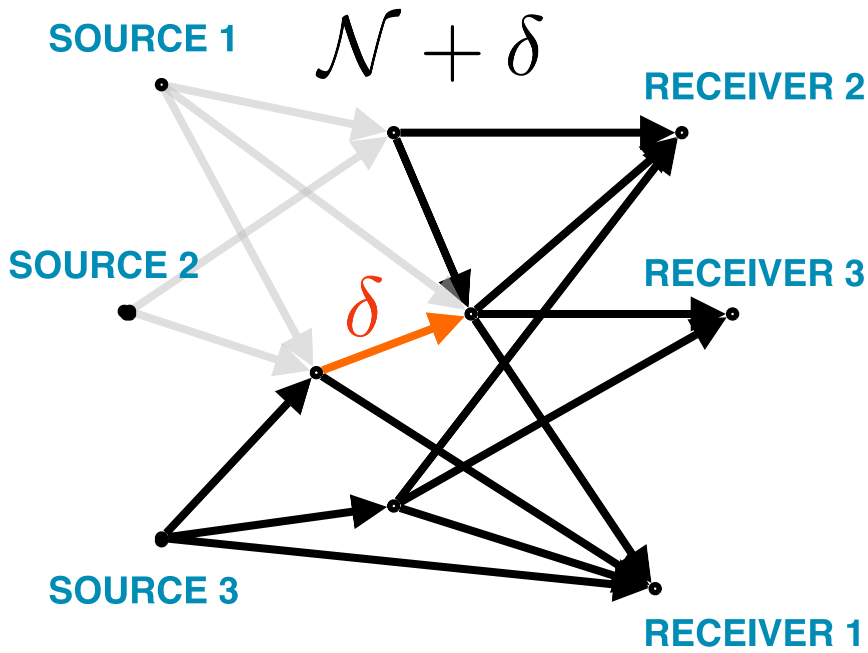

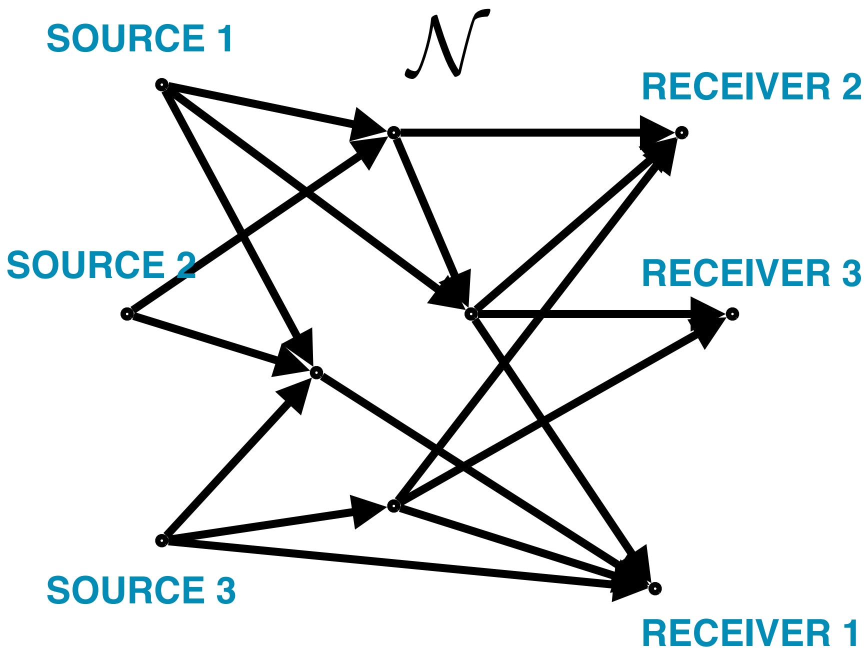

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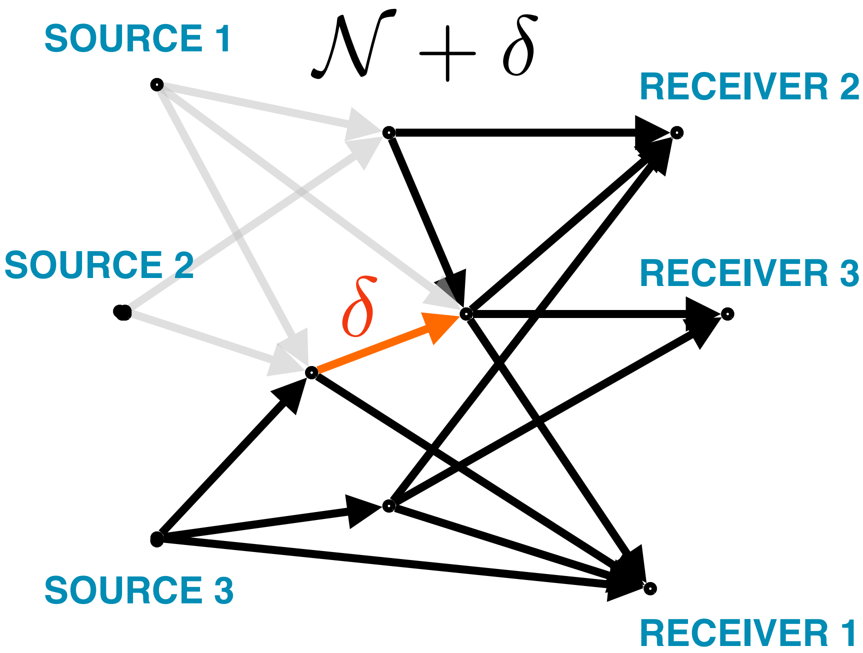

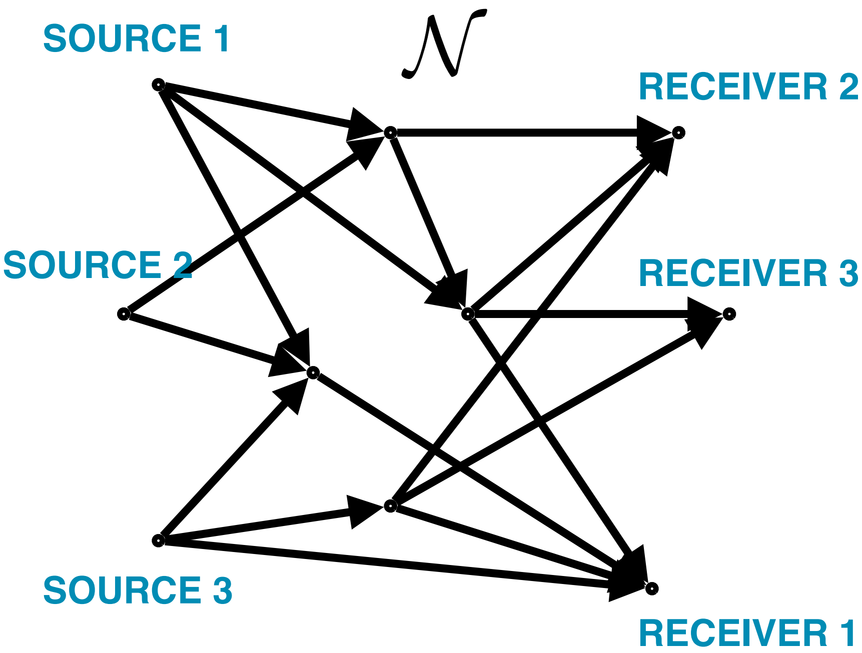

$$\text{Is Capacity}(\mathcal{N} + \delta) \subseteq \text{Capacity}(\mathcal{N}) + [0, \delta]^k?$$

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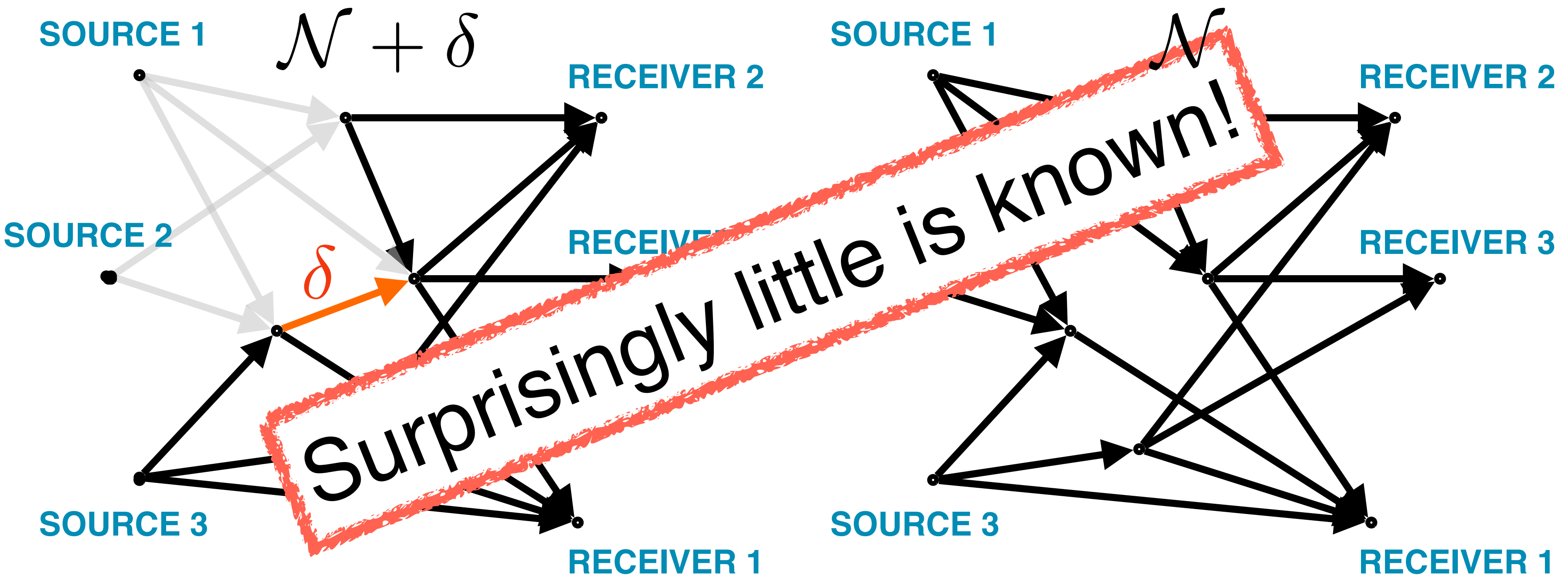


$$\text{Is Capacity}(\mathcal{N} + \delta) \subseteq \text{Capacity}(\mathcal{N}) + [0, \delta]^k?$$

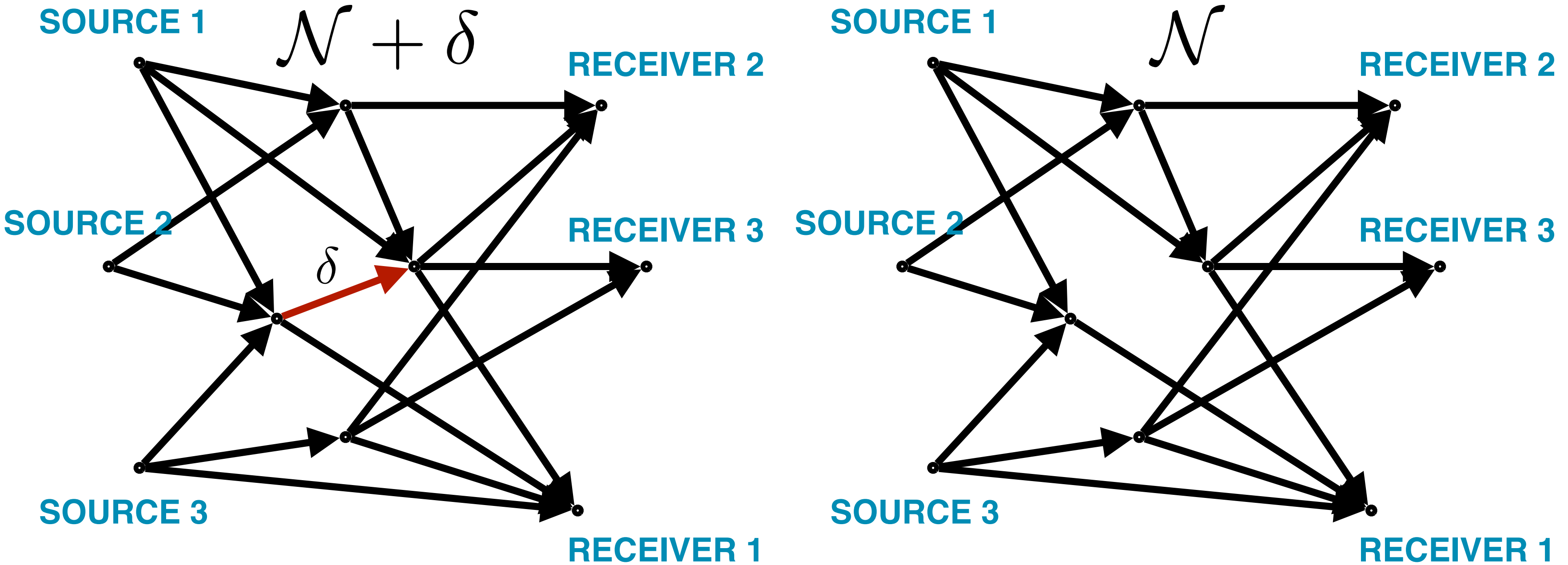
The question remains unsolved in wireline networks:

[Jalali, Effros, Ho 2011, 2012, Langberg, Effros 2012, Lee, Langberg, Effros 2013]

- The impact of a link is bounded by its capacity in some networks:
  - if cut-set bounds are tight (e.g., single- & multi-source multicast)
  - for co-located sources, super-source networks, terminal edges
  - for linear codes, “separable” codes
  - in index coding.
- The same property holds in current outer bounds:
  - cut-set bounds
  - Generalized network sharing bounds [Kamath, Tse, Anantharam 2011]
  - Generalized Linear Programming (LP) bound [Yeung 1997, Song, Yeung 2003]
- No proof that this property always holds.
- No examples where this property fails.
- Equivalence to other problems (0- vs. ϵ-error, dep srcs, NC vs. IC, …)

Wireline networks: Intuition

[Jalali, Effros, Ho 2011, 2012, Langberg, Effros 2012, Lee, Langberg, Effros 2013]



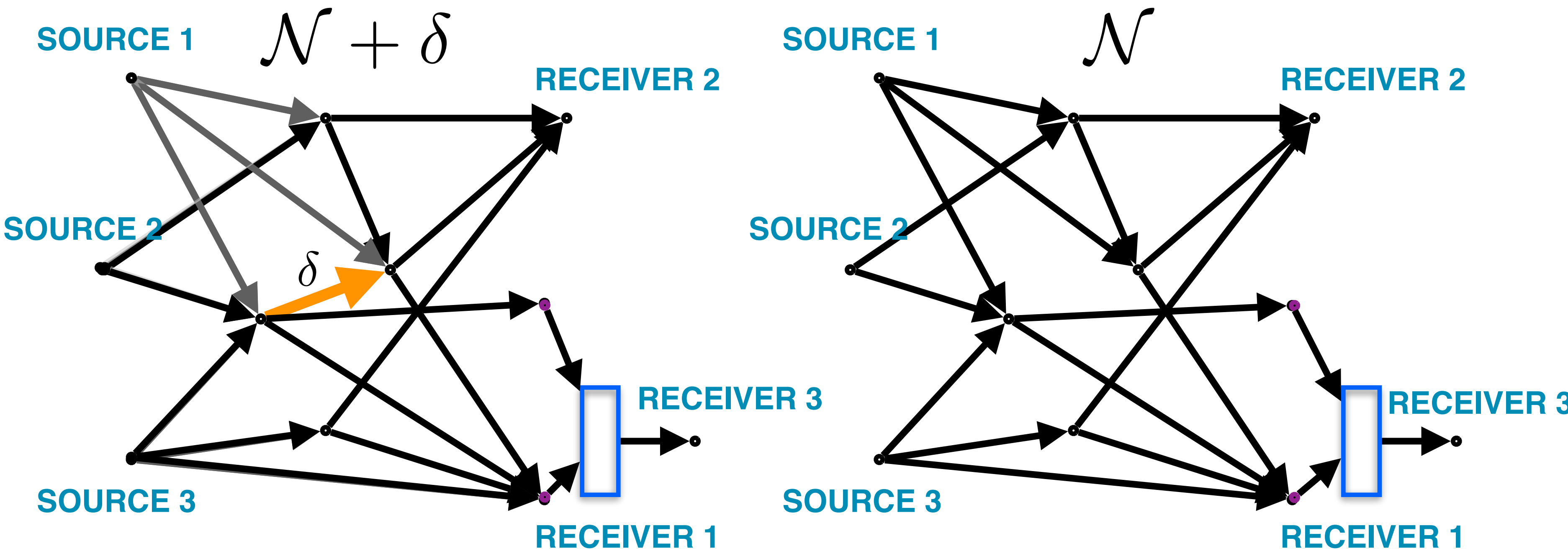
Only send source values that give the most common transmission across our connection.

The number of such transmissions supports rate

$$(R_1 - \delta, R_2 - \delta, R_3 - \delta)$$

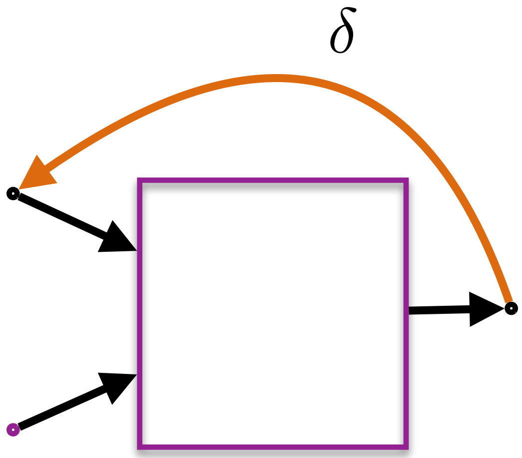
Challenge: This strategy may not always be possible.

What happens in networks containing wireless components?

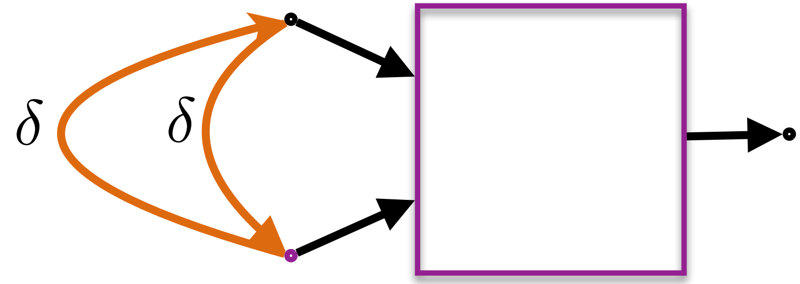


$$\text{Is Capacity}(\mathcal{N} + \delta) \subseteq \text{Capacity}(\mathcal{N}) + [0, \delta]^k?$$

In prior literature, the benefit of any edge was bounded by the capacity of that edge.



[Sarwate & Gastpar 2009]



[Willems 1983]

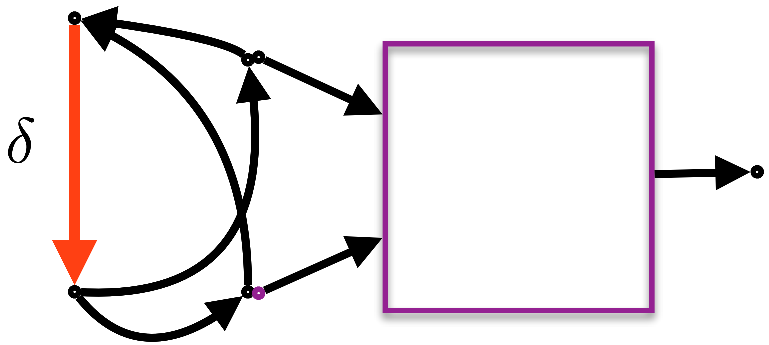
$$\text{Is Capacity}(\mathcal{N}+\delta) \subseteq \text{Capacity}(\mathcal{N}) + [0,\delta]^k?$$

YES.

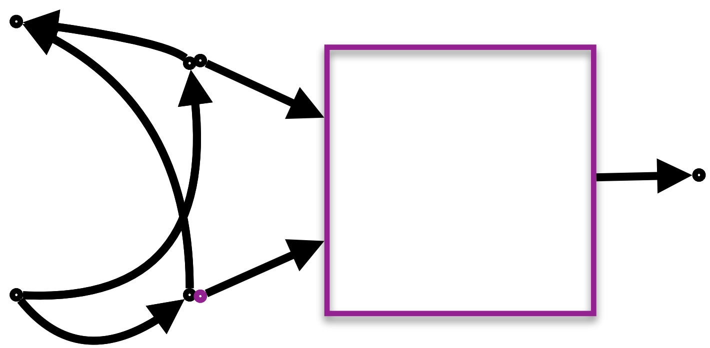
For general memoryless networks, the benefit of an edge can exceed its capacity.

[Noorzad, Effros, Langberg, Ho 2014]

$\mathcal{N} + \delta$



$\mathcal{N}$

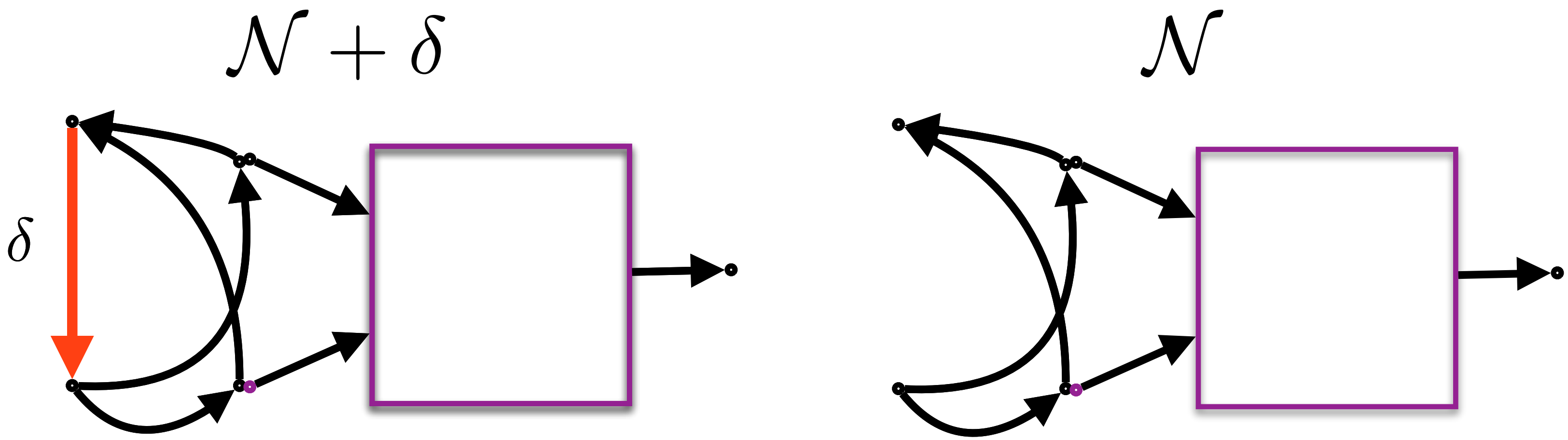


$$\text{Is Capacity}(\mathcal{N} + \delta) \subseteq \text{Capacity}(\mathcal{N}) + [0, \delta]^k?$$

NO.

By how much can the benefit of an edge exceed its capacity?

[Noorzad, Effros, Langberg, Ho 2014]

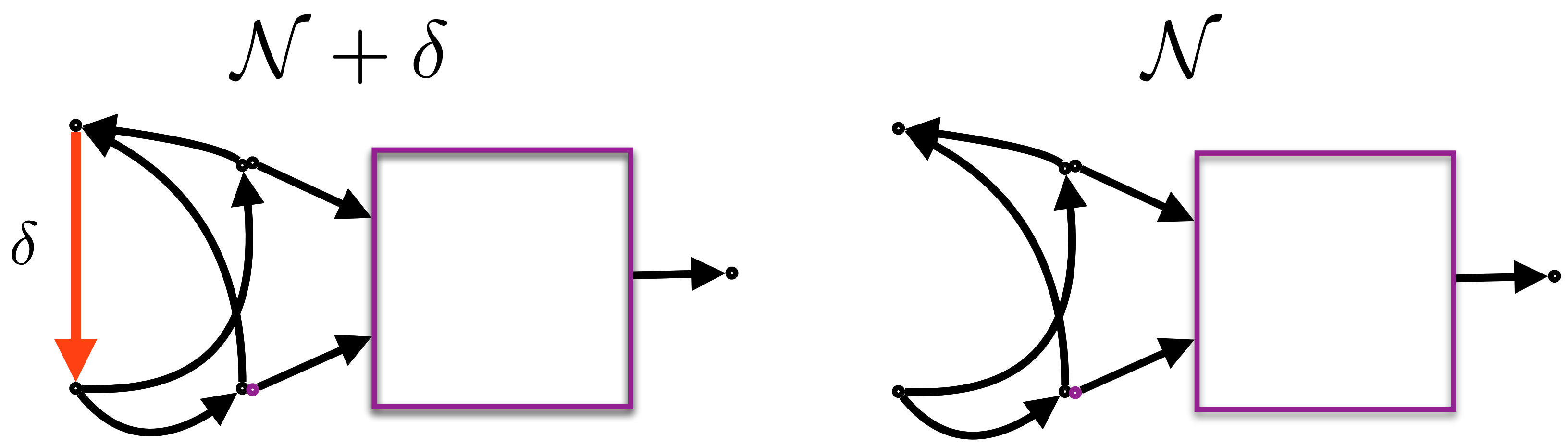


$$\text{Capacity}(\mathcal{N} + \delta) \subseteq \text{Capacity}(\mathcal{N}) + [0, f(\delta)]^k ?$$

NO!!! (for ANY polynomial f)

The benefit of an edge can FAR exceed its capacity!

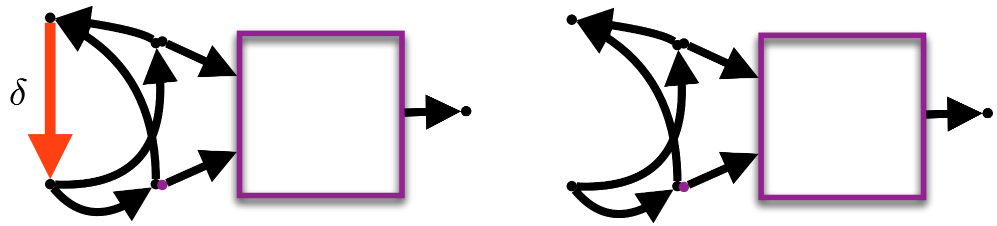
[Noorzad, Effros, Langberg, Ho 2014]



Adding a δ-capacity link can increase the network capacity ALMOST EXPONENTIALLY in δ.

The benefit of an edge can FAR exceed its capacity!

[Noorzad, Effros, Langberg, Ho 2014]



$$\mathcal{X}_1 = \mathcal{X}_2 = \{1, \ldots, M\}$$

$$\mathcal{Y} = (\mathcal{X}_1 \times \mathcal{X}_2) \cup \{E\} \text{ (}E\text{ denotes "erasure")}$$

$$G = \begin{bmatrix} g(1,1) & g(1,2) & \ldots & g(1,M) \\ g(2,1) & g(2,2) & \ldots & g(2,M) \\ \vdots & \vdots & \ddots & \vdots \\ g(M,1) & g(M,2) & \ldots & g(M,M) \end{bmatrix}$$

$$p(y|x_1,x_2) = \begin{cases} 1(y = (x_1,x_2)) & \text{if } g(x_1,x_2) = 1 \\ 1(y = E) & \text{if } g(x_1,x_2) = 0 \end{cases}$$

Proof (counter-example) [Noorzad, Effros, Langberg, Ho 2014]

$$G = \begin{bmatrix} g(1,1) & g(1,2) & \dots & g(1,M) \\ g(2,1) & g(2,2) & \dots & g(2,M) \\ \vdots & \vdots & \ddots & \vdots \\ g(M,1) & g(M,2) & \dots & g(M,M) \end{bmatrix}$$

$$p(y|x_1,x_2) = \begin{cases} 1(y=(x_1,x_2)) & \text{if } g(x_1,x_2)=1 \\ 1(y=E) & \text{if } g(x_1,x_2)=0 \end{cases}$$

Set $\delta = 2\log\log M$

$\exists$ G such that:

$\exists$ $\frac{M}{\log M \log\log M}$-partition of $\mathcal{X}_1$ ($\mathcal{X}_2$) s.t. each "cell" contains a "1"

Ensures $\mathcal{C}(\mathcal{N}+\delta)$ large ($R_1 + R_2 = 2\log M - O(\log\log M)$) ach)

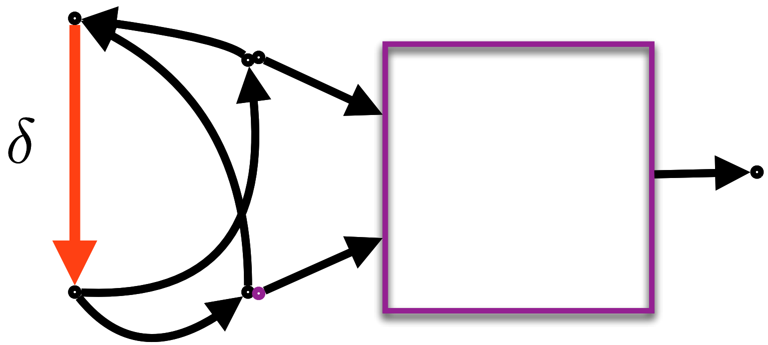
Every sufficiently large sub-matrix has fraction $\geq 1-\epsilon$ "0"s

Ensures $\mathcal{C}(\mathcal{N})$ small ($R_1 + R_2 < 1.25\log M$)

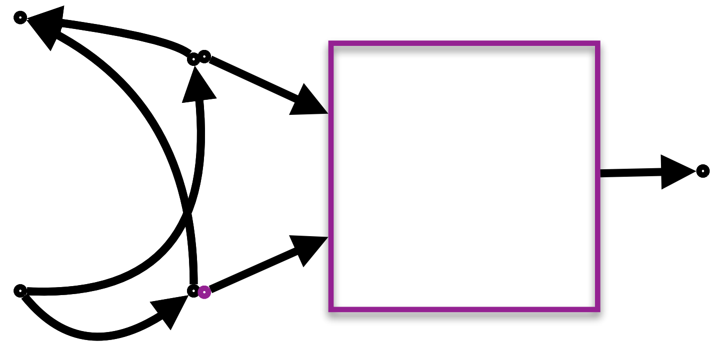
The benefit of an edge can far exceed its capacity.

[Noorzad, Effros, Langberg, Ho 2014]

$\mathcal{N} + \delta$



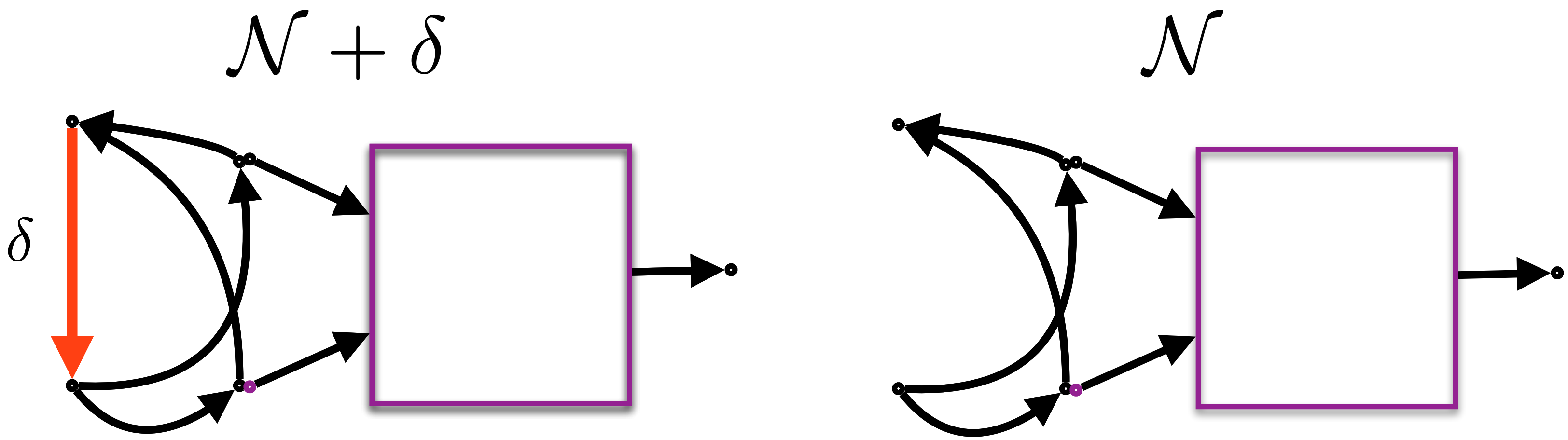
$\mathcal{N}$



But this is an artificial example...

What happens in more realistic networks?

[Noorzad, Effros, Langberg 2015]



If $C(\mathcal{N} + \delta)|_{\delta=\infty} \supsetneq C(\mathcal{N})$ AND

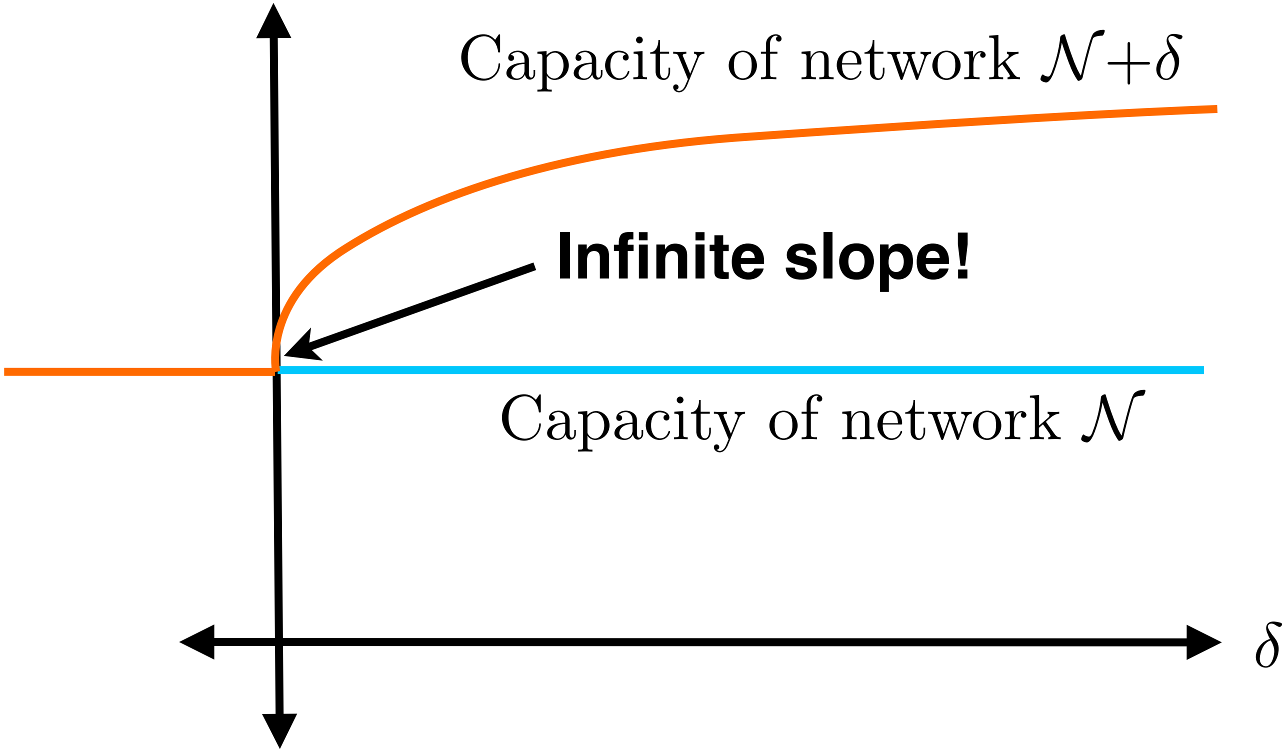
f(δ) is the smallest function for which

$$C(\mathcal{N} + \delta) \subseteq C(\mathcal{N}) + [0, f(\delta)]^k$$

then $\frac{d}{d\delta} f(\delta)|_{\delta=0} = \infty$

If cooperation helps at all, then a little cooperation can help a LOT!

[Noorzad, Effros, Langberg 2015]



Can rate-0 cooperation ever help???

[Noorzad, Effros, Langberg 2016]

Surprisingly, at least in
the case of maximal-error capacity,
the answer is YES!

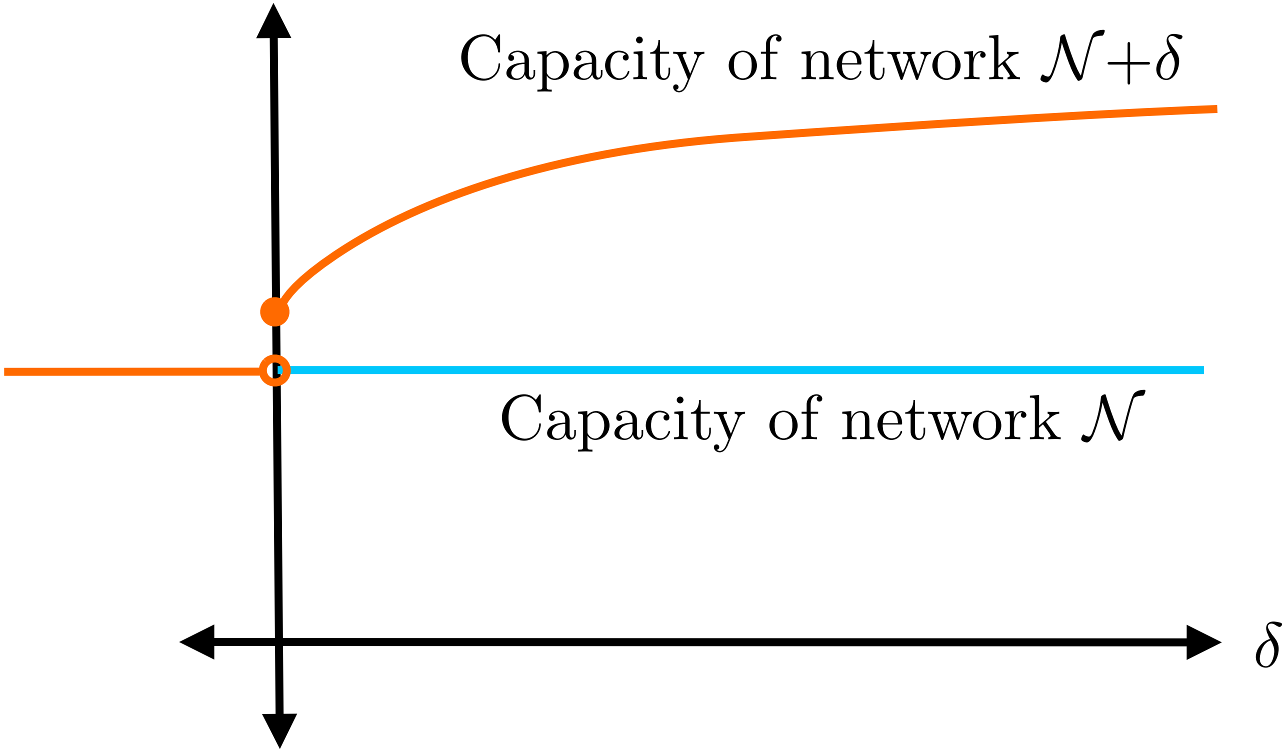
There exist networks where

$f(\delta)$

is discontinuous at $\delta=0$.

The curve can be discontinuous.*

[Noorzad, Effros, Langberg 2016]



** In the max-error case.*

Can 1 bit of cooperation ever help?

Can 1 bit of cooperation ever help?

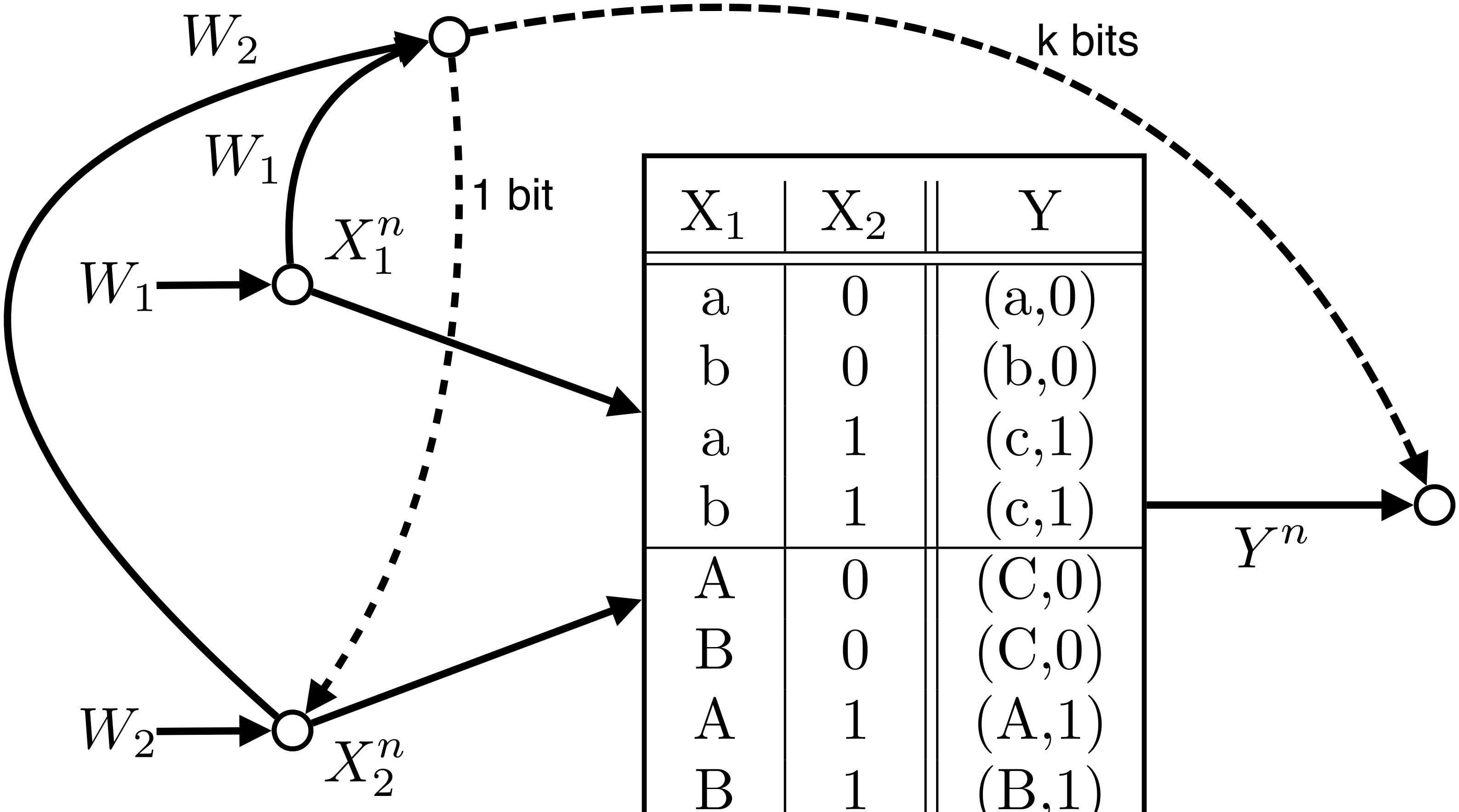
[Langberg, Effros 2016]

YES!

There exists a network where
1 bit of cooperation
changes the maximal-error network capacity!

How can 1 bit of cooperation help???

[Langberg, Effros 2016]



X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

[Dueck 1978]

Dueck MAC:

X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

t	1	2	3	4	5	6	7	8
$X_{1,t}$	A	a	B	B	b	A	B	A
$X_{2,t}$	0	1	1	0	1	1	1	0
Y_t	(C,0)	(c,1)	(B,1)	(C,0)	(c,1)	(A,1)	(B,1)	(C,0)

$X_{1,t}$	A	a	B	B	b	A	B	A
$X'_{2,t}$	1	0	0	1	0	0	0	1
Y_t	(A,1)	(a,0)	(C,0)	(B,1)	(b,0)	(C,0)	(C,0)	(A,1)

Dueck MAC:

X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

t	1	2	3	4	5	6	7	8
$X_{1,t}$	A	a	B	B	b	A	B	A
$X_{2,t}$	0	1	1	0	1	1	1	0
Y_t	(C,0)	(c,1)	(B,1)	(C,0)	(c,1)	(A,1)	(B,1)	(C,0)

$X_{1,t}$	A	a	B	B	b	A	B	A
$X'_{2,t}$	1	0	0	1	0	0	0	1
Y_t	(A,1)	(a,0)	(C,0)	(B,1)	(b,0)	(C,0)	(C,0)	(A,1)

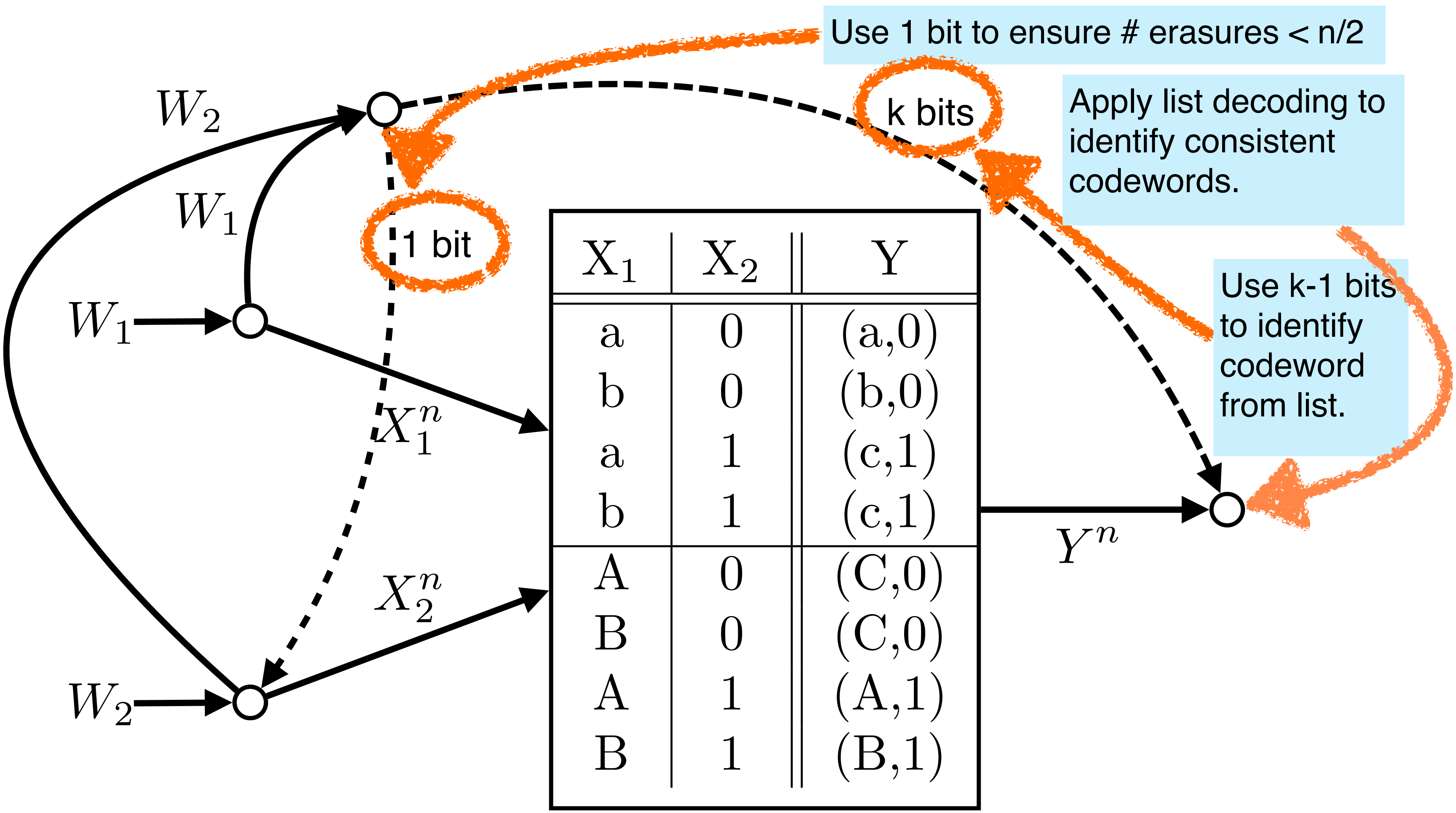
Dueck MAC:

X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

t	1	2	3	4	5	6	7	8
$X_{1,t}$	A	a	B	B	b	A	B	A
$X_{2,t}$	0	1	1	0	1	1	1	0
Y_t	(C,0)	(c,1)	(B,1)	(C,0)	(c,1)	(A,1)	(B,1)	(C,0)
	↑	↑	↓	↑	↑	↓	↓	↑
$X_{1,t}$	A	a	B	B	b	A	B	A
$X'_{2,t}$	1	0	0	1	0	0	0	1
Y_t	(A,1)	(a,0)	(C,0)	(B,1)	(b,0)	(C,0)	(C,0)	(A,1)

How can 1 bit of cooperation help???

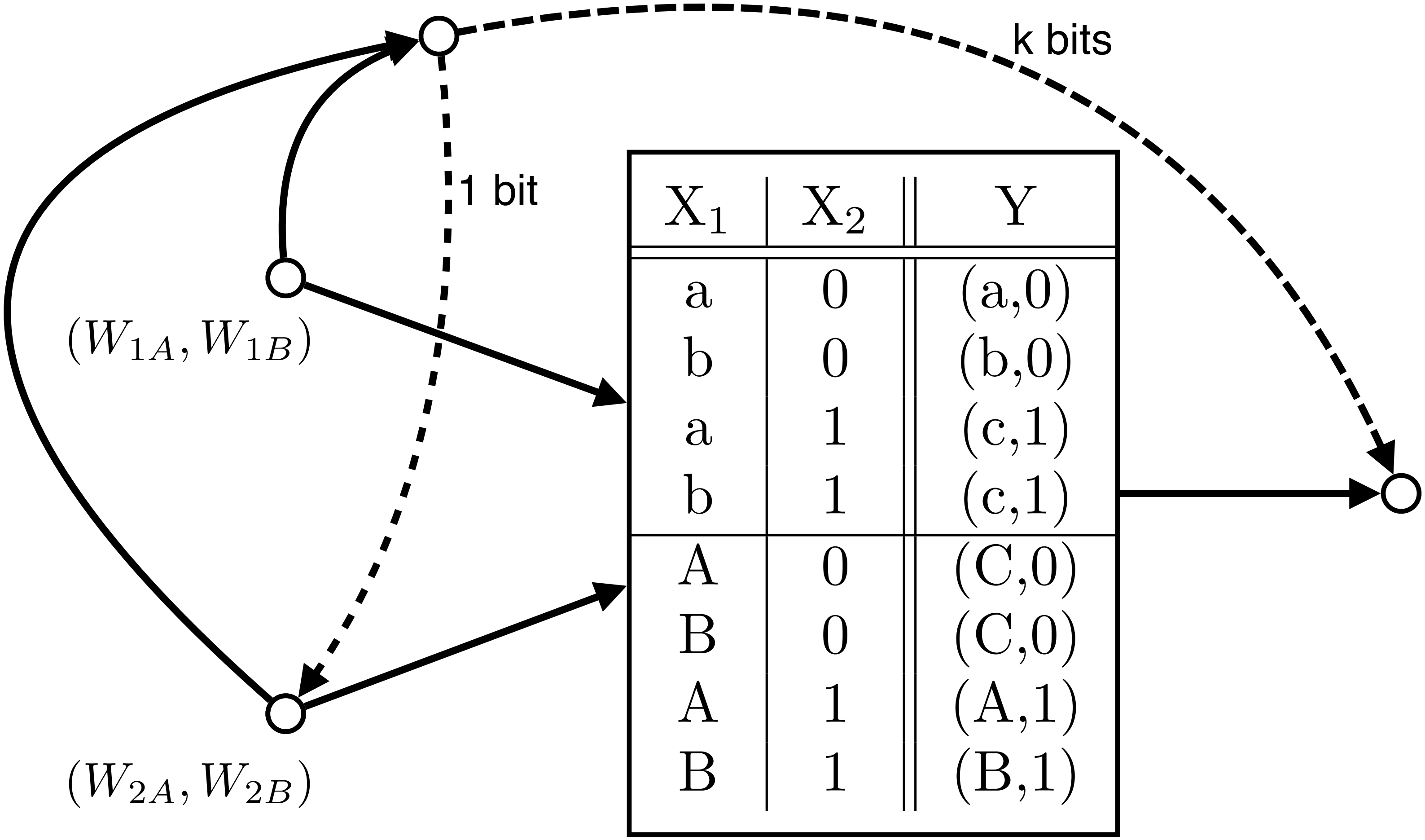
[Langberg, Effros 2016]



X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

[Dueck 1978]

Blocklength-(2n+1) strategy



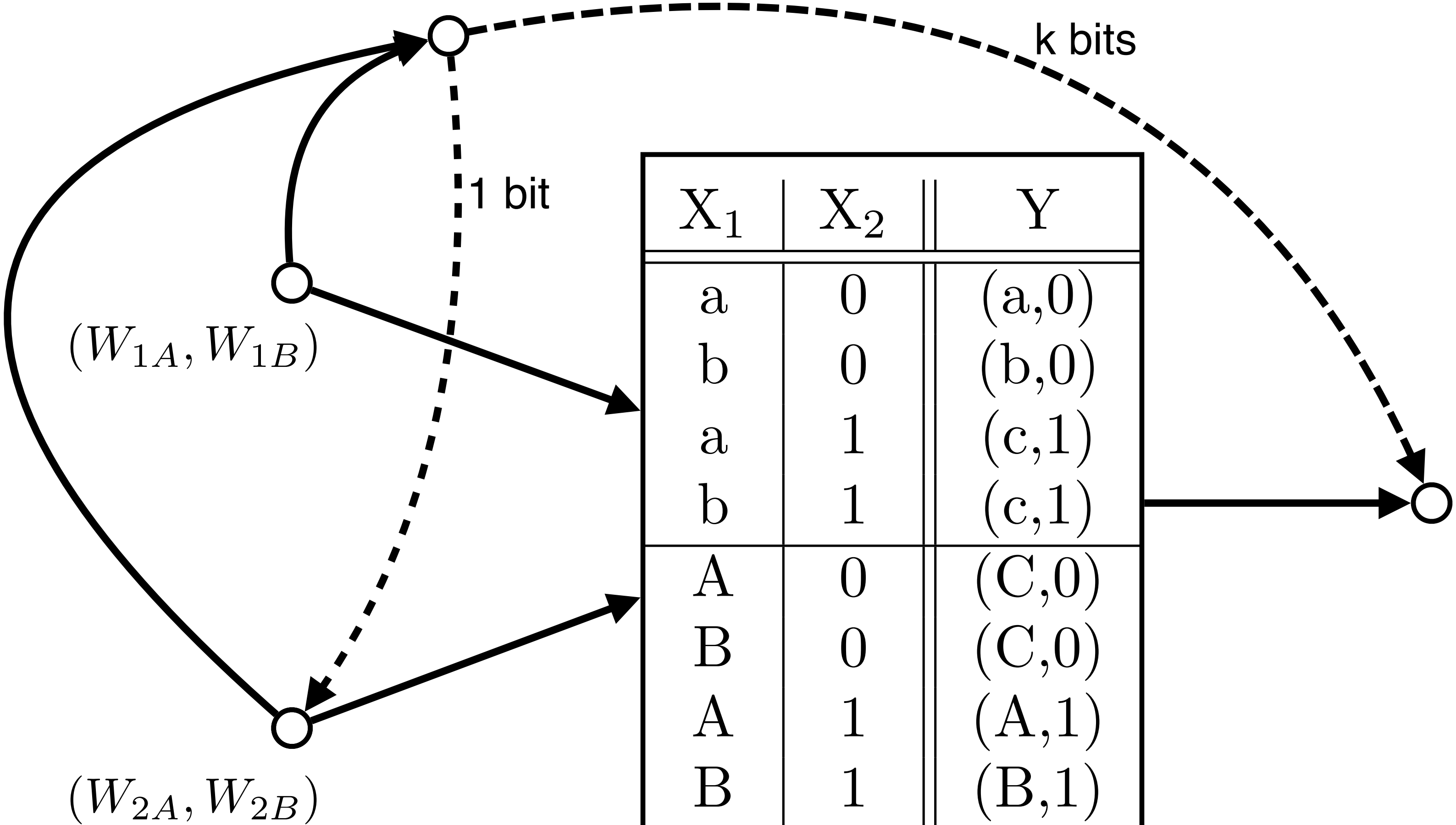
X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

[Dueck 1978]

Blocklength-(2n+1) strategy

$(R_{1,A}, R_{2,A}) \in \mathcal{C}(\text{Dueck})$

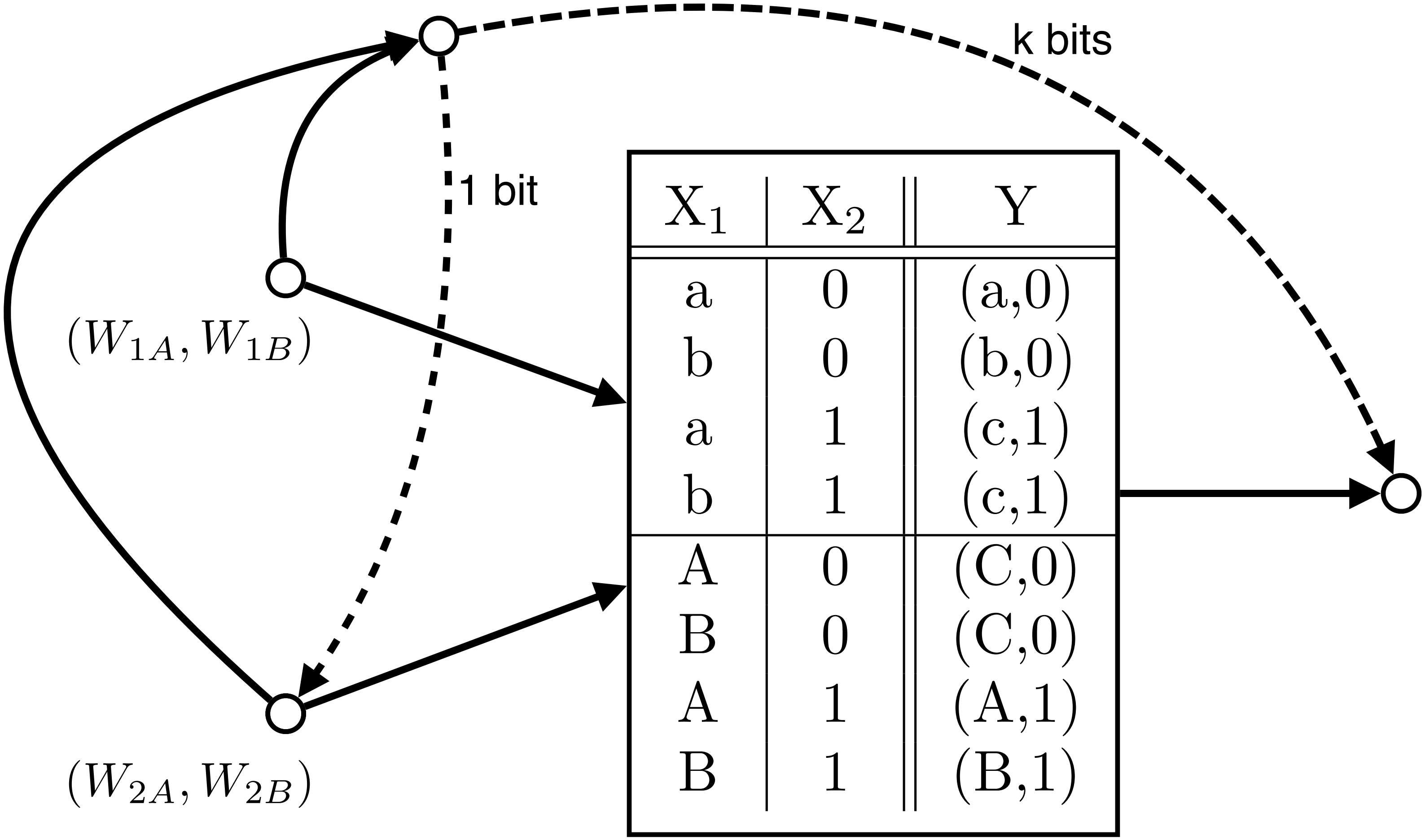
$(R_{1,B}, R_{2,B}) \notin \mathcal{C}(\text{Dueck})$



X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

[Dueck 1978]

Blocklength-(2n+1) strategy

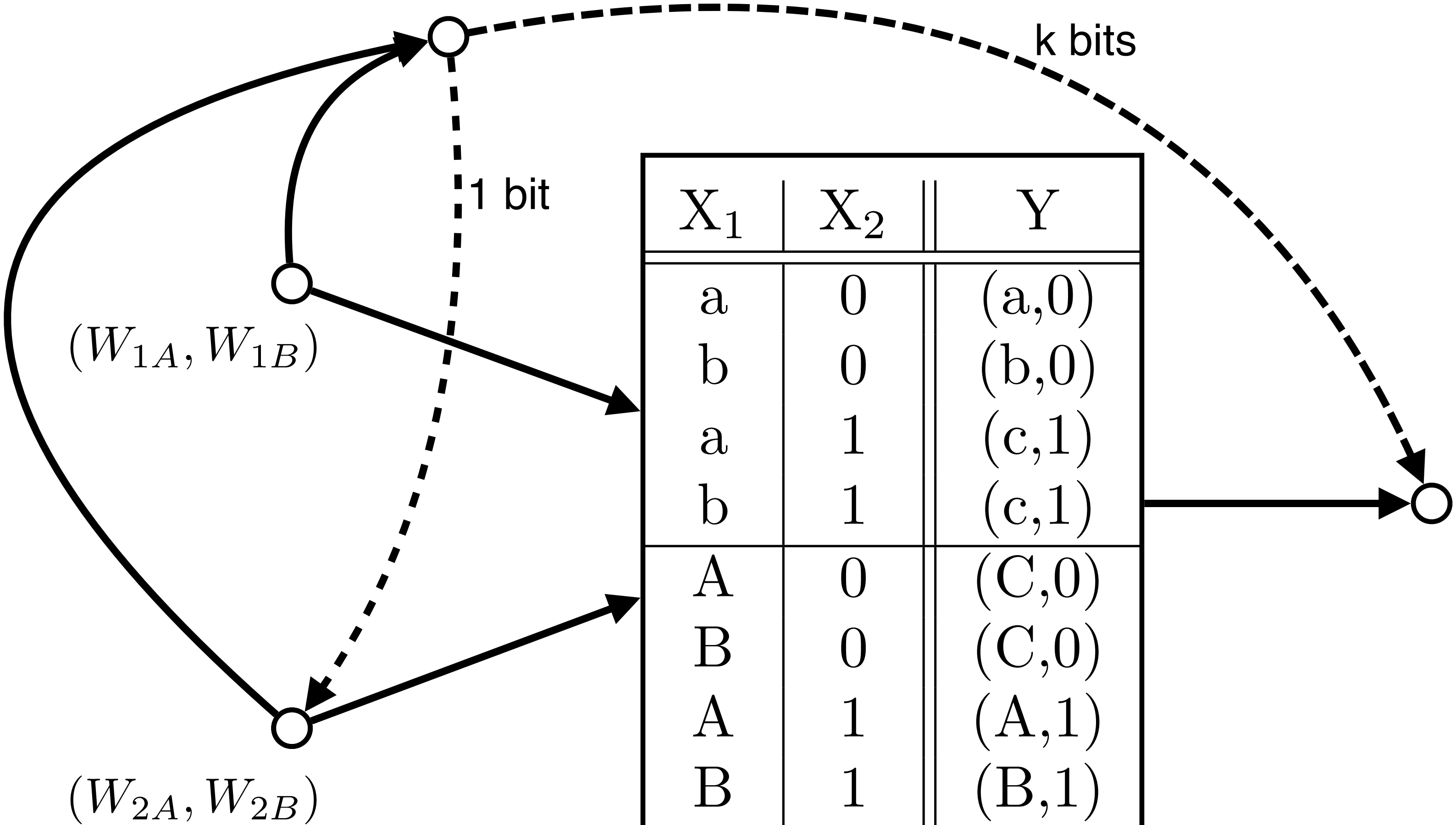


X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

[Dueck 1978]

Blocklength-(2n+1) strategy

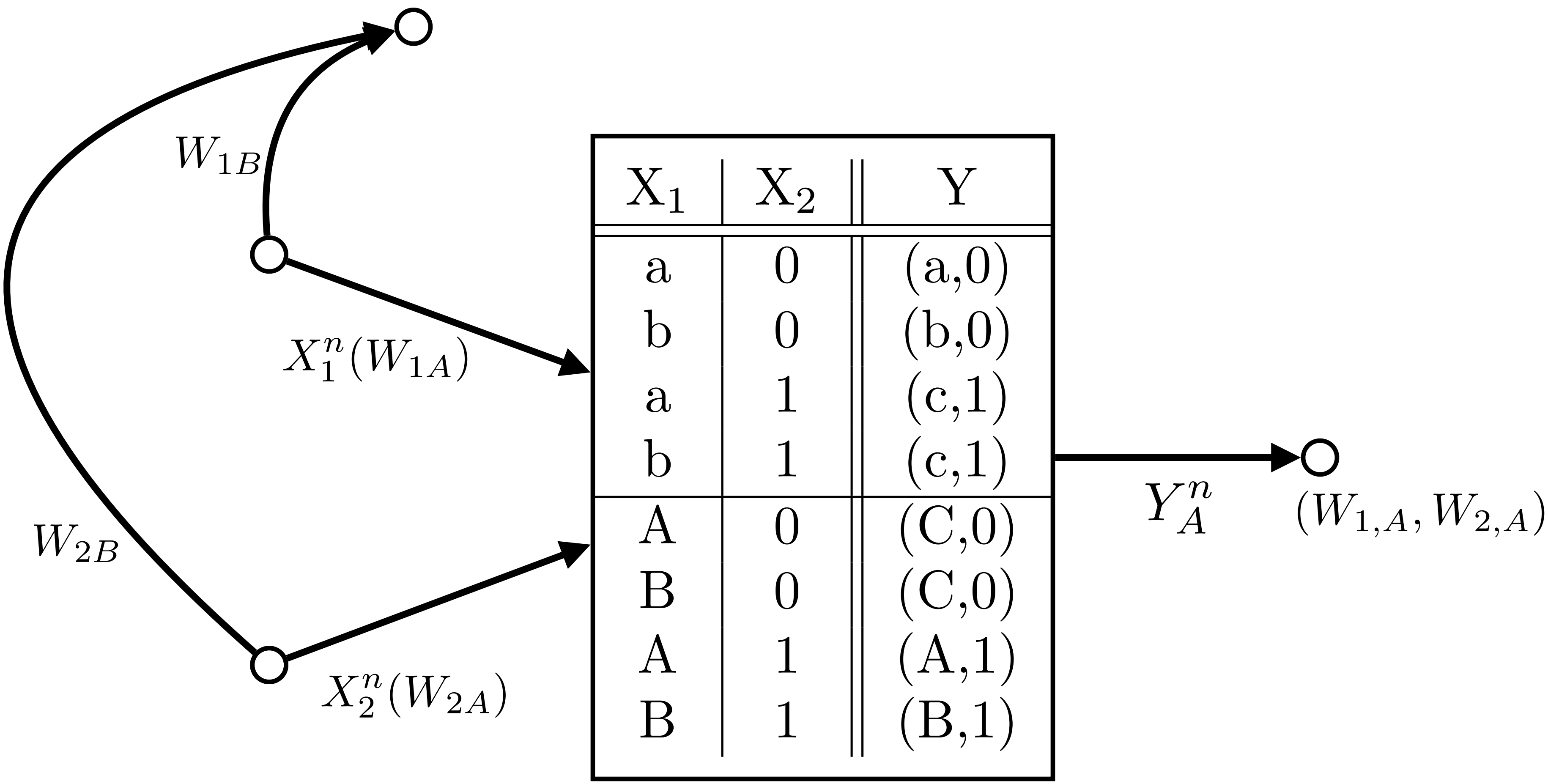
$$\left(\frac{R_{1,A}+R_{1,B}}{2}, \frac{R_{2,A}+R_{2,B}}{2}\right) \notin \mathcal{C}(\text{Dueck})$$



X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

[Dueck 1978]

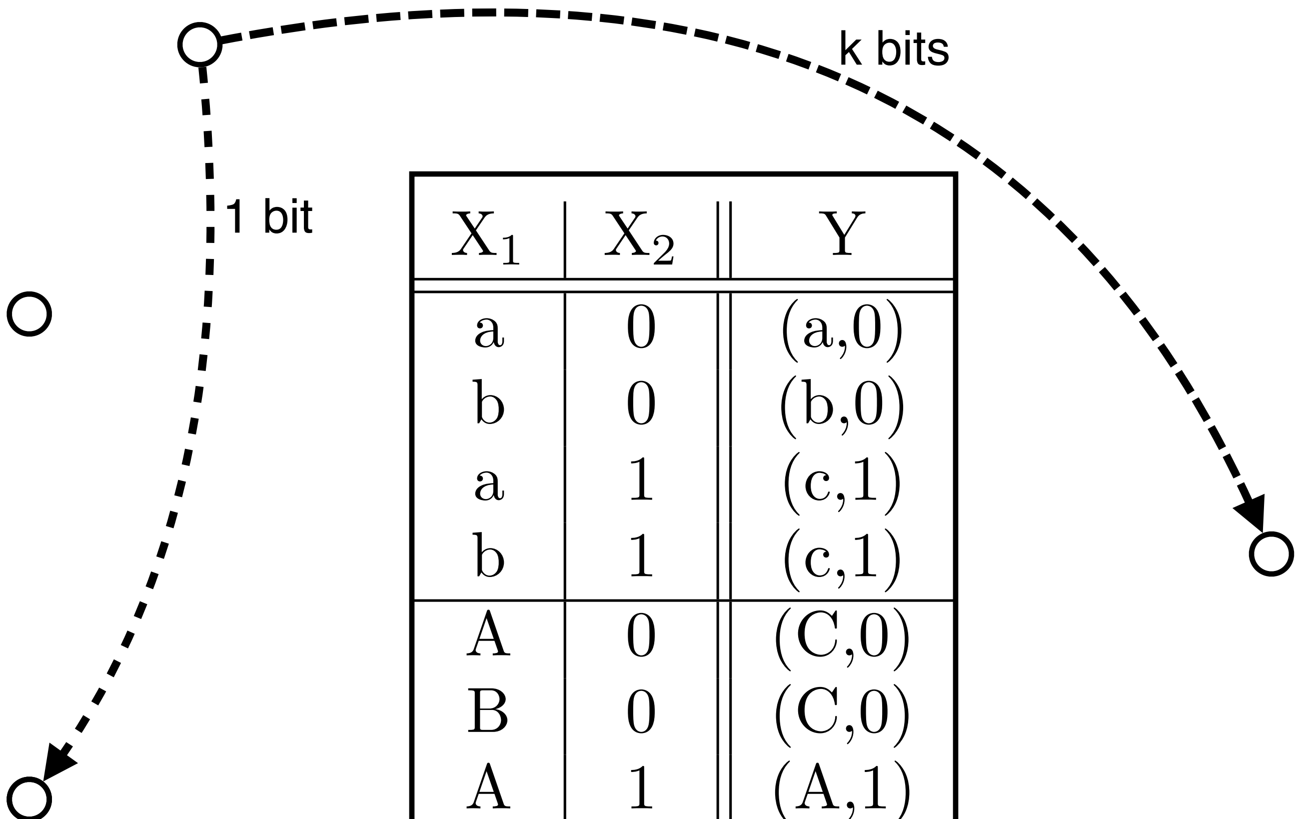
Time steps 1,...,n



X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

[Dueck 1978]

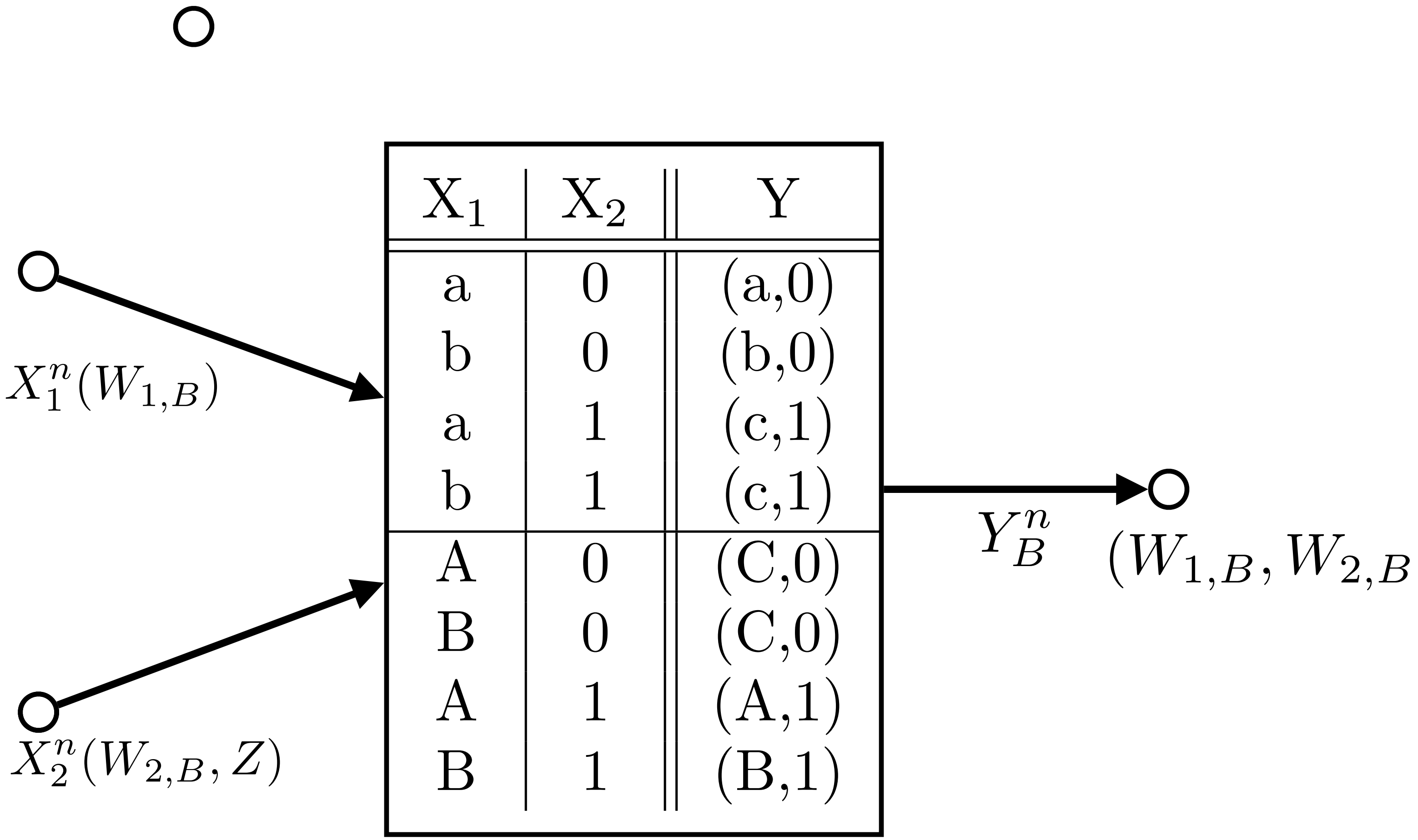
Time step n+1



X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

[Dueck 1978]

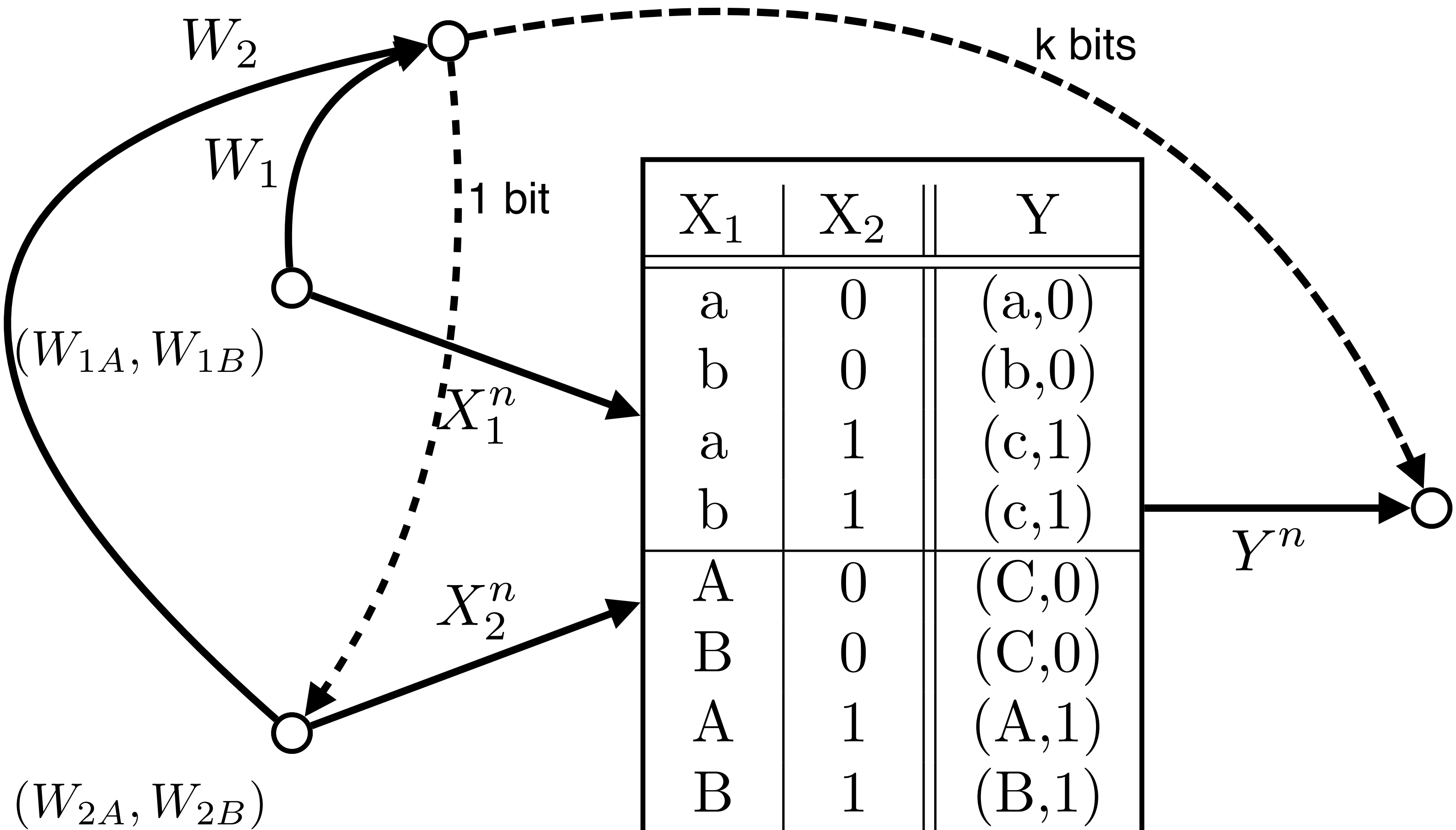
Time steps n+2,...,2n+1



X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

[Dueck 1978]

Blocklength-(2n+1) strategy

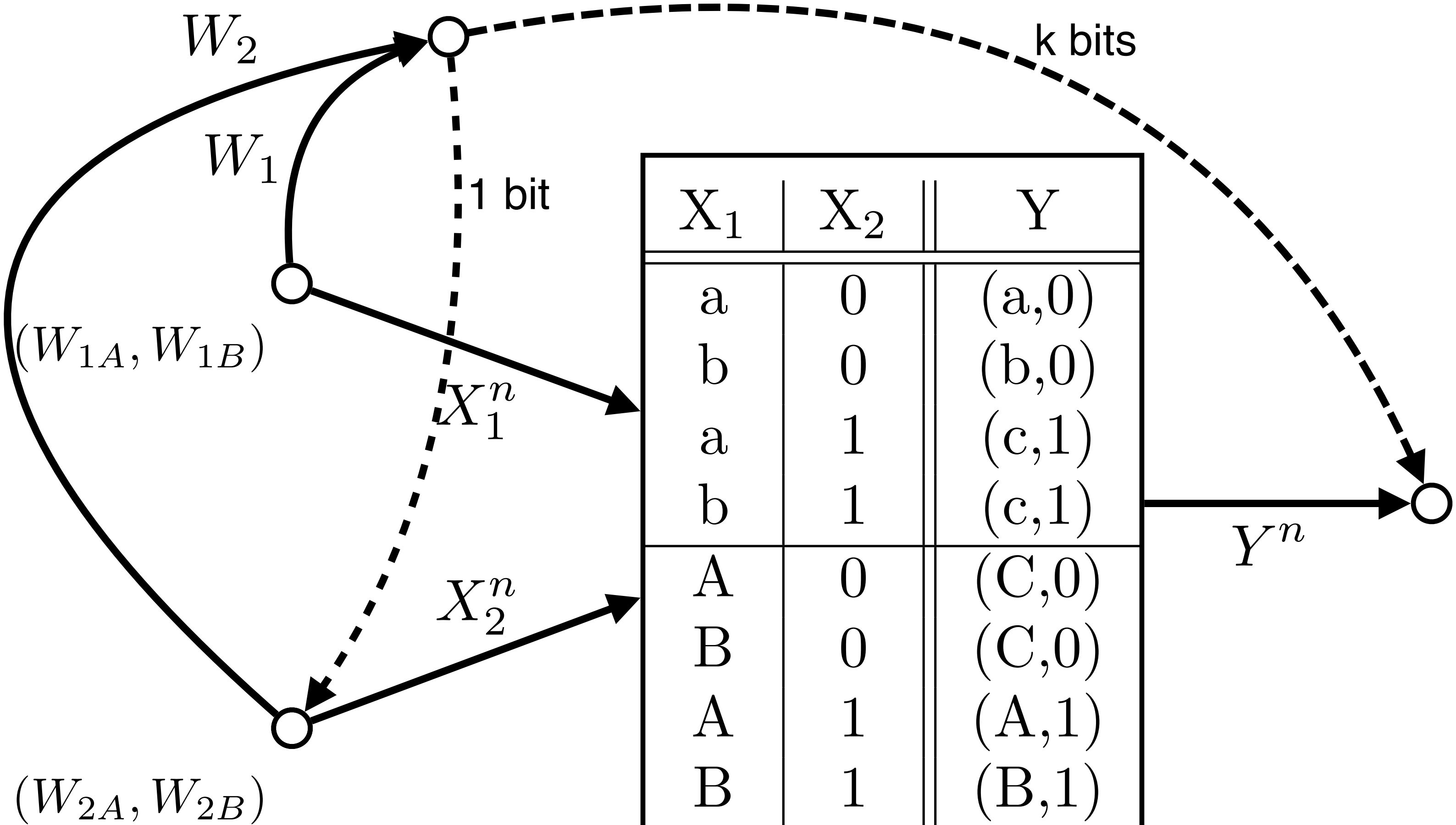


X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

[Dueck 1978]

Blocklength-(2n+1) strategy

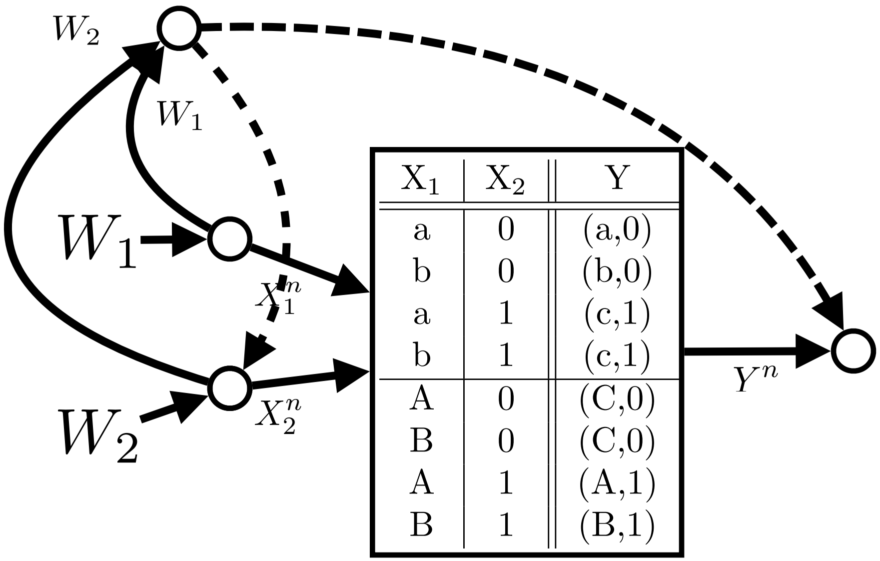
$$\left(\frac{R_{1,A}+R_{1,B}}{2}, \frac{R_{2,A}+R_{2,B}}{2}\right) \notin \mathcal{C}(\text{Dueck})$$



X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

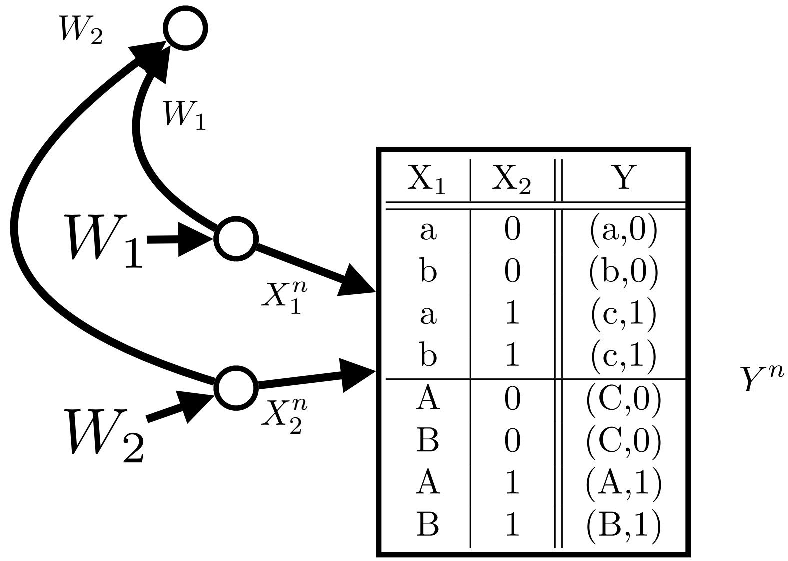
[Dueck 1978]

(k+1) bits can change the network capacity. Can 1 bit change network capacity?



X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

$\mathcal{N}^+$



X_1	X_2	Y
a	0	(a,0)
b	0	(b,0)
a	1	(c,1)
b	1	(c,1)
A	0	(C,0)
B	0	(C,0)
A	1	(A,1)
B	1	(B,1)

$\mathcal{N}$

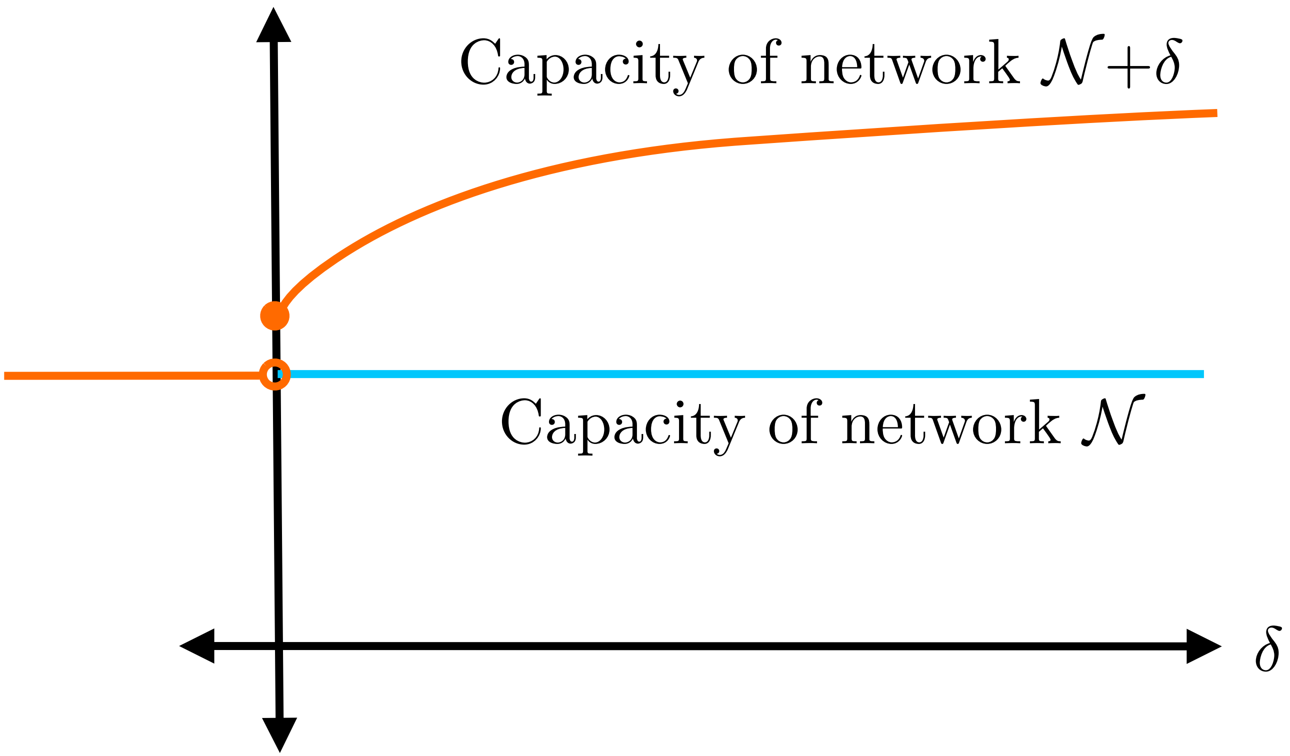
YES!

$$\text{Capacity}(\mathcal{N}^+) \neq \text{Capacity}(\mathcal{N})$$

$$C(\mathcal{N}^+) = C(\mathcal{N}_{k+1}) \supseteq C(\mathcal{N}_k) \supsetneq \cdots \supseteq C(\mathcal{N}_1) \supseteq C(\mathcal{N}_0) = C(\mathcal{N})$$

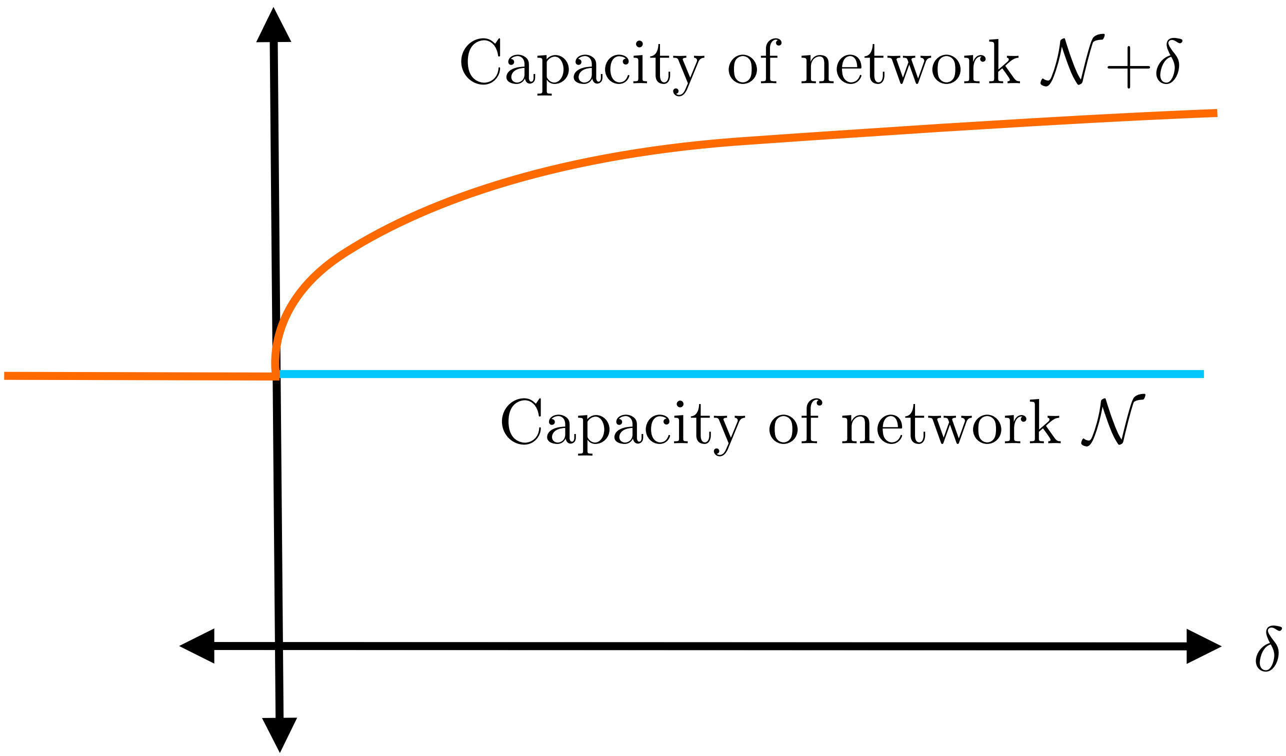
Even 1 bit can yield a discontinuity!

[Langberg, Effros 2016]



** In the max-error case.*

Is a discontinuity possible in the average error case???



[Noorzad, Effros, Langberg, 2018]

Summary

- The differential view of network capacity bounds the impact of a channel in a larger network.
- For wireline networks, it is unknown whether the benefit of a single edge can ever exceed its capacity.
  - In some cases, it provably cannot.
  - Current outer bounds likewise suggest that it cannot.
  - The question is related to other interesting unsolved questions.
- For networks with wireless connections, the impact of a single edge can FAR exceed its capacity.
  - Gap can be large.
  - Slope can be infinite.
  - Rate-0 links can help.
  - Even 1 bit can help!